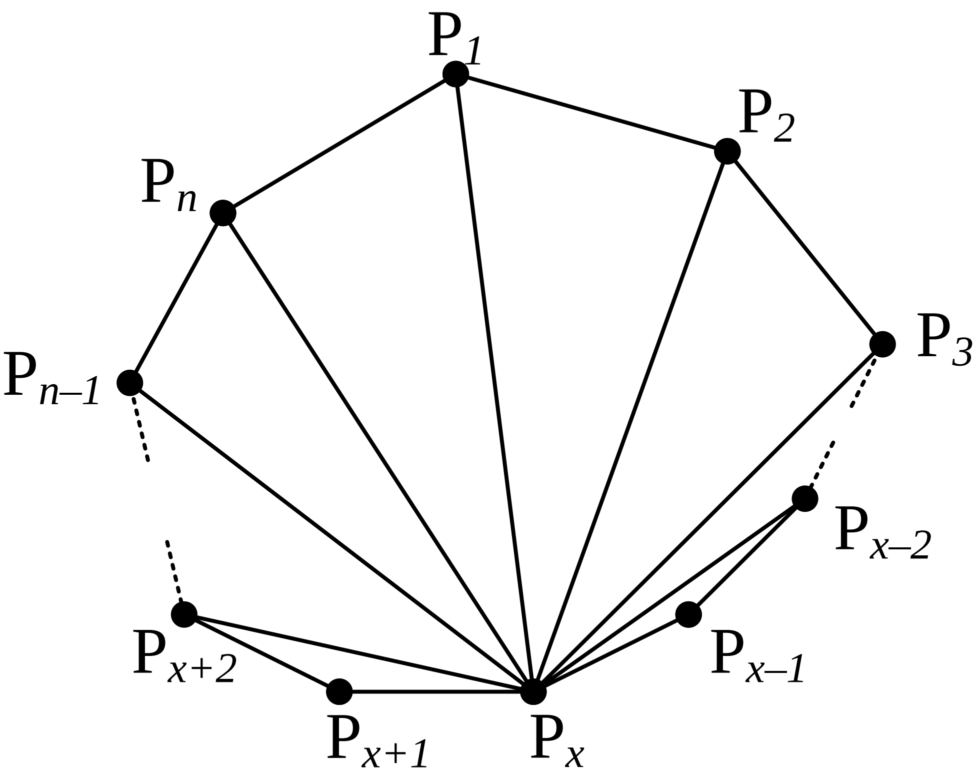
Ardhi Prabowo

Basic GEOMETRY

Book 1 : Polygon



Ardhi Prabowo

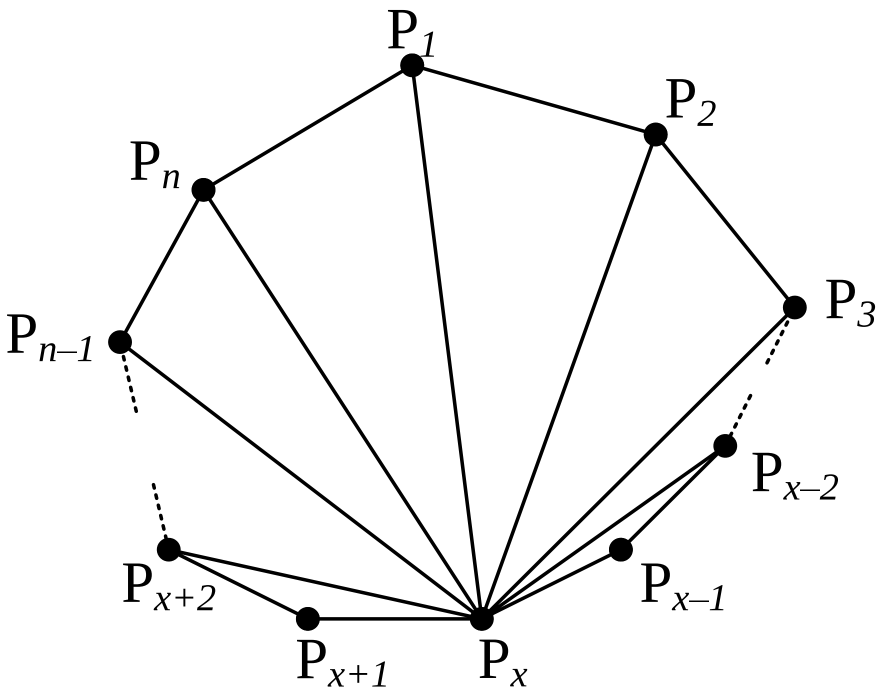
Handout

Basic

GEOMETRY

Book 1: Polygon

For international student on Mathematics Education



International Standard School Teacher Education
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2010

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Mathematics Classification Data

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Introduction

Alhamdulillah, thanks to Allah, here we finish to arrange the text book of basic geometry book 1. In this book we only discuss about properties of polygon that is triangle until finite n-gon. We start from basic knowledge of geometry until the complex formula.

We introduce structure of this book in its hierarchy. We usually begin with fact (that is axioms) and derivationing to definition. From definition, we arrange the properties by theorems or its corollary. We often proof theorems; however we also leave some proof as exercise to the reader. We never prove axioms and definition because its fact.

This book was arranged for whole topics. You maybe only need some topics; so that you can choice topics you need to learn. We suggest you to learn whole of book content, because each topics has relation to topics before.

Here we give some analytics proofing beside geometric proofing only. So that the proof is stronger. We also give reason when proofing a theorem, such that the reader could trace back the proof. It is an additional benefit for the reader.

Some subjects materials are given in unique way, that is using exercise. We give enough exercise to reader. We wish by doing the exercise, the reader should get additional information about the topics.

We're gratefully thanks to Bunda Kusni and Serge Lang for all ideas. We need critics from the reader such that this book become better.

November, 2010.

Regards,
Ardhi

General Overview

A. Competences Standard

After learning this book the student should be:

1. understood the principal of geometry.
2. using the principal of geometry to solve problems

B. Basic Competence

The student should be:

1. understood the principal of geometry,
2. understood the locus of geometry object in plane,
3. understood the characteristics of triangle,
4. understood the characteristics of Polygon,
5. able to proof the area of polygons and Pythagoras Theorem,
6. understood the next parallel law,
7. understood the next triangle laws, and
8. understood the law of circle.

C. Prequisite

Prequisite material required is as follows:

1. English language of Mathematics
2. Algebra

D. Method of Learning Handout

The reader of this book are expected to learn with steps as follows:

1. Read carefully the contents of this textbook.
2. Listen carefully to explanation lecturer.
3. Ask the lecturer if nor clear.
4. Doing all the task or exercises that existed at this textbook.
5. Develop by own this material by reading and stusying book related to the basic geometry.

Contents

Chapter 5
THE AREA AND PYTHAGORAS THEOREM

Chapter 6
THE NEXT PARALLEL LAW

Chapter 7
HIGHER LEVEL TRIANGLE

Chapter I

P R E   G E O M E T R Y

We start to learn basic geometry. In this whole subject material, we will learn about drawing, analysis, and logic. Some theorems are proved by its contraposition and it need logic to understand the proof. In first chapter we will discuss about the principal of learning geometry, axiom, definition, theorem, and the base of drawing.

A. AXIOM, DEFINITION, AND PRE-PRINCIPAL

Some relation of elements in geometry sometimes must be accepting without proof. This relation called axioms or postulate. Relation which must be proofed is called theorem or dale. The difference between postulate and axioms that is axiom obtain in general science but postulate obtain in specific science.

Axiom example:
"Through 2 points, we just can draw 1 line."

We introduce the word 'iff'. Iff is short word from if and only if. We usually used these words to explain any similarity to two sentences. Example, "an isosceles triangle is a triangle which has three similar legs" could be written as "a triangle is called isosceles triangle iff has three similar legs."

Definition is also unproved. The difference between definition and axioms, definition is specific to a material. Example:

Definition of Grup:
"Let G any set.
G grup iff G suitable for:

1. There are binary operation * in G, that is closed.
2. There is identity element on G, which is e.
3. For every $x \in G$ there exist $x' \in G$, such that $x * x' = e$."

B. LINE AND ANGLE

1. Line (Basic Properties of Line)

The geometry presented in this course deals mainly with figures such as points, lines, triangles, circles, etc., which we will study in a logical way. We begin by briefly and systematically stating some basic properties. For the moment, we will be working with figures which lie in a plane. You can think of a plane as a flat surface which extends infinitely in all directions. We can represent a plane by a piece of paper or a blackboard.

Axiom.
1.1 Given two distinct points P and Q in the plane, there is one and only one line which goes through these points.

We denote this line by L_{PQ} or sometimes denoted by small caps g. We have indicated such a line in Figure below. The line actually extends infinitely in both directions.

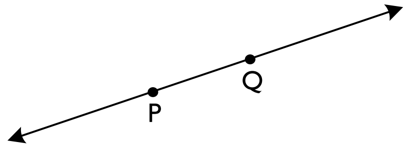


Figure 1.1. Ray of PQ

We define the line segment, or segment between P and Q, to be the set consisting of P, Q and all points on the line L_{PQ} lying between P and Q. We denote this segment by $\overline{PQ}$.

If we choose a unit of measurement (such as the inch, or centimeter, or meter, etc.) we can measure the length of this segment, which we denote d(P, Q). If the segment were 5 cm long, we would write d(P, Q) = 5 cm. Frequently we will assume that some unit of length has been fixed, and so will write simply d(P, Q) = 5, omitting reference to the units. We also write d(P, Q) by $|PQ|$.

Two points P and Q also determine two rays, one starting from P and the other starting from Q, as shown in Figure 1.2. Each of these rays starts at a particular point, but extends infinitely in one direction.

Thus we define a ray starting from P to consist of the set of points on a line through P which lie to one side of P, and P itself. We also say that a ray is a half line. The ray starting from P and passing through another point Q will be denoted by R_{PQ}. Suppose that Q' is another point on this ray, distinct from P. You can see that the ray starting from P and passing through Q is the same as the ray that starts from P and passes through Q'.

Using our notation, we would write

$$R_{PQ} = R_{PQ'}$$

In other words, a ray is determined by its starting point and by any other point on it. The starting point of a ray is called its **vertex**.

Sometimes we will wish to talk about lines without naming specific points on them; in such cases we will just name the lines with a single letter, such as K or L. We define lines K and L to be **parallel** if either K = L, or K ≠ L and K does not intersect L. Observe that we have allowed that a line is parallel to itself. Using this definition, we can state three important properties of lines in the plane.

Axiom

1.2 Two lines which are not parallel meet in exactly one point.

1.3 Given a line L and a point P, there is one and only one line passing through P, parallel to L.

In Figure (a) below we have drawn a line k passing through P parallel to m. In Figure (b) we have drawn a line k which is not parallel to m, and intersects at a point Q.

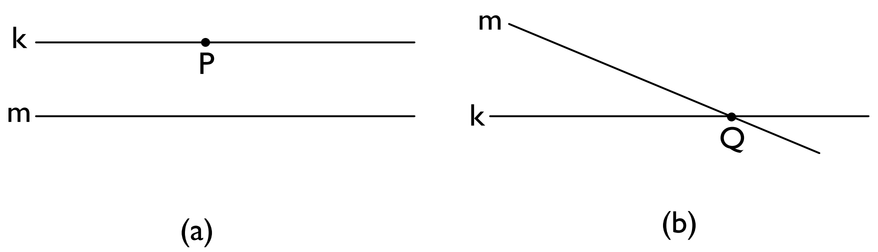


Figure 1.2. Condition of two lines in a plane

> **Axiom!**
> 1.4 Let L_1, L_2, and L_3 be three lines. If L_1 is parallel to L_2 and L_2 is parallel to L_3, then L_1 is parallel to L_3.

This property is illustrated in Figure.

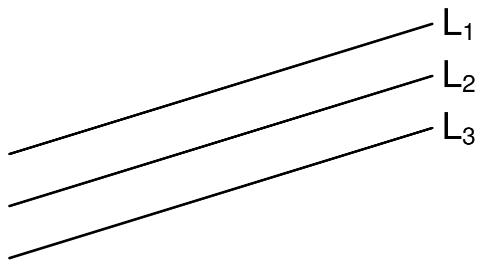


Figure 1.3. Basic condition of parallel lines in a plane

We define two segments, or two rays, to be parallel if the lines on which they lie are parallel. To denote that lines L_1 and L_2 are parallel, we use the symbol

$$\mathbf{L_1 \parallel L_2}$$

Note that we have assumed properties axiom. That is, we have accepted them as facts without any further justification. Such facts are called axioms or postulates. Information about parallel line will be explained at chapter 2 and chapter 6.

2. Angle (Basic Properties of Angle)

a. Angle Notation
   Look at the figure below.

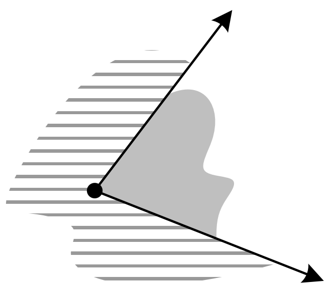


Figure 1.4. Angle

An angle can build from two rays. When two rays meet at one point, it will be held two angles. The solid region is the region of **minor angle**, and the shaded region is the region of **major angle**. It seems like when we talk about circle.

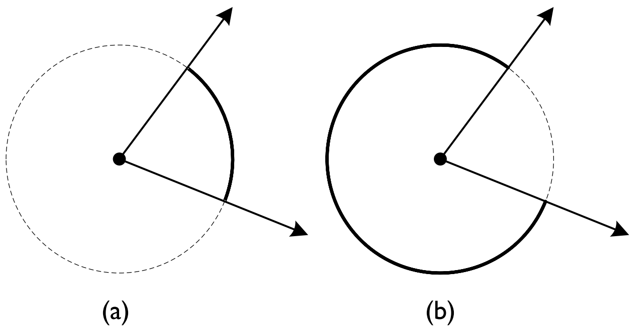


Figure 1.5. Angle on circle

Figure 1.5 (a) show us angle which corresponding to minor arc of a circle, the we usually call it minor angle, so do major angle (*see figure 1.5 (b)*). The major angle is the angle which is corresponding the major arc of a circle.

b. Part of Angle

As another object of geometry, angle also has its parts. We will discuss what the part of angle is. Look at the figure 1.6 below.

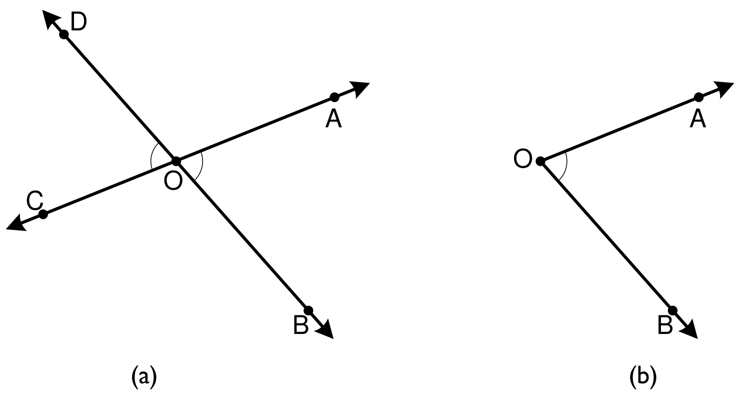


Figure 1.6. Angle and part of angle

We can see that if two rays intersect each other, there will be 4 angles lie on the rays (*see figure 1.6 (a)*). For next discussion, if there is no information indeed, we may assume that **an angle is a minor angle**.

Now please give attention to figure 1.6 (b). Point O is usually called by **Angle Point**. OA and OB is usually called by **legs** or legs of angle. This angle is usually called by angle O ($\angle$O) or angle AOB ($\angle$AOB) or angle BOA ($\angle$BOA). We write the measure or the size of angle O by **m$\angle$O**.

Note that the size of angle in this topic is never negative. We will explain about the negative value for size of angle later in coordinate topics.

c. Kinds of Angle

Based on the size of angle, we may classify angle into 4 kinds of angle, those are:
1. Acute angle is angle that has size less than 90°, see figure 1.7 (a).
2. Obtuse angle is angle that has size more than 90°, see figure 1.7 (b).
3. Right angle is angle that has size 90°, see figure 1.7 (c).
4. Straight angle is the angle that has size 180°, see figure 1.7 (d).

To see the drawn of the angle above, please see figure 1.7 below.

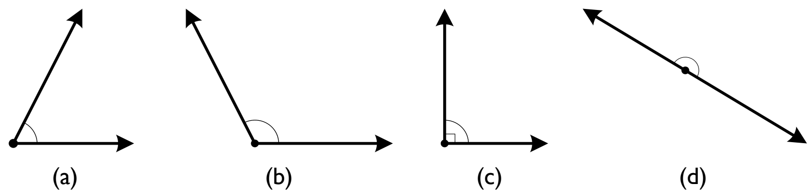


Figure 1.7. Kinds of angle

d. Condition of angle.

As figure 1.6 (a) we may understand that if two rays intersect each other, there will be hold 4 angles. The condition of angle that could explain the situation are:
1) supplementary,
2) complementary,
3) opposite angle, and
4) adjacent angle.

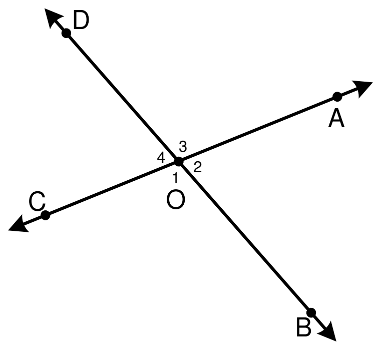


Figure 1.8. The condition of angle

$\angle O_1$and $\angle O_2$ is called **supplementary adjacent angle**, the consequence is $\angle O_1$ become supplement of $\angle O_2$ and vice versa. $\angle O_1$ and $\angle O_3$ is called **opposite angle**.

Definition:

1.1 Supplementary Angle.

Let $\angle A$ any angle. $\angle B$ is supplementary angle if $m\angle A + m\angle B = 180^\circ$, in other word we say that $\angle B$ is addition angle for $\angle A$ to be 180°.

1.2 Complementary Angle.

Let $\angle C$ any angle. $\angle D$ is complementary angle if $m\angle C + m\angle D = 90^\circ$, in other word we say that $\angle D$ is addition angle for $\angle C$ to be 90°.

1.3 Opposite Angle.

2 angles are called 2 opposite angle if has similar angle point and both of the legs make a straight line.

1.4 Adjacent Angle.

2 angles are called 2 adjacent angles if has one common leg and has similar angle point.

Basic properties of line and angle we end by the theorems below. We will explain more detail in the next chapter.

Theorems!

1.1 The difference between supplementary and complementary similar angle is 90°
1.2 Two angles that has similar supplementary angle, has similar size.
1.3 Two angles that has similar complementary angle, has similar size.
1.4 The opposite angle has similar size.

Proof!

1.1 Prove that The difference between supplementary and complementary similar angle is 90°

Let $\angle A$ is any angle.
Let $\angle B$ is complementary angle of $\angle A$.
Obvious $m\angle A + m\angle B = 90^\circ$
$\Leftrightarrow m\angle A + m\angle B + 90^\circ = 180^\circ$
$\Leftrightarrow m\angle A + m\angle D = 180^\circ$, for some $m\angle D = m\angle B + 90^\circ$.
We get $\angle D$ is supplementary angle of $\angle A$.
Obvious $m\angle D = m\angle B + 90^\circ$
$\Leftrightarrow m\angle D - m\angle B = 90^\circ$.
So the difference between supplementary and complementary similar angle is 90°.

1.2 Two angles that has similar supplementary angle, has similar size.

Let $\angle A$ and $\angle B$ any set.
Let $\angle C$ is supplementary angle of $\angle A$ and $\angle B$.
Obvious $m\angle A + m\angle C = 180^\circ$ and $m\angle B + m\angle C = 180^\circ$ also.
Obvious $m\angle A + m\angle C = m\angle B + m\angle C$ and by subtracting the size of $\angle C$ to each side we get $m\angle A = m\angle C$.
So two angles that it supplementary angle is same size, has similar size.

1.3 Two angles that has similar complementary angle, has similar size.

Let $\angle A$ and $\angle B$ any set.
Let $\angle C$ is complementary angle of $\angle A$ and $\angle B$.

Obvious m∠A + m∠C = 90° and m∠B + m∠C = 90° also.
Obvious m∠A + m∠C = m∠B + m∠C and by subtracting the size of ∠C to each side we get m∠A = m∠C.
So two angles that it complementary angle is same size, has similar size.

1.4 The opposite angle has similar size.

Look at figure 1.3
Obvious A_1 and A_3 are opposite angle.
Obvious A_2 is supplementary angle of A_1 and A_2 is also supplementary angle of A_3.
By theorem 1.2 it is proved that $m\angle A_1 = m\angle A_3$.
So the opposite angle has similar size.

C. THE BASIC OF DRAWING

Some of subject material could not explain theoretically. To make the reader understand about the specific subject material, we make new structure that is construction. We need the reader to increase their activity when do this construction. The tools that are needed are only compass, 2 triangle ruler, pencils, and drawing book. Prepare it well!

1. Construction 1: Dividing a line segment into two similar parts.

Sometimes in geometry we choose special vertex to make the reader easy to understand. Usually we choose midpoint to make reader understand about the principal of vertex in a line, without subtracting the concept, or maybe the midpoint became the question of a theorem.

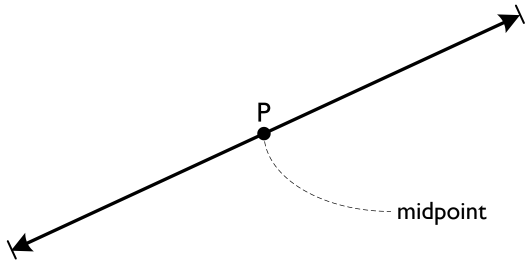


Figure 1.9. The midpoint

The principal to find the mid point is making perpendicular bisector to the segment given. The steps are explained directly by solving the question given. By solving the question, we wish the reader get more information and could increasing their skill to draw geometrically.

Question:
Find the midpoint of segment AB below.

Figure 1.10.

Answer:
By using the compass and point A as center of arc, make an arc which has radius more than a half of $|AB|$. *Note that* $|AB|$ *means the length of* AB. See figure 1.11 below.

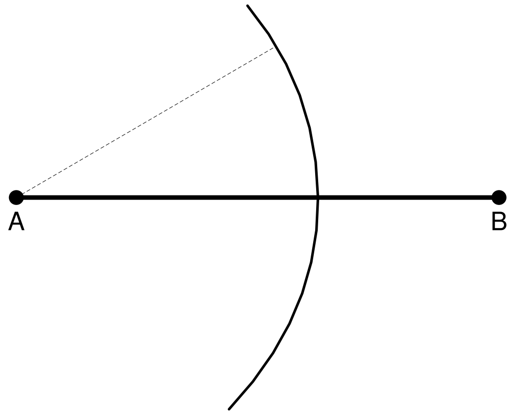


Figure 1.11. Step one

Next step is just like before, but we move the center of arc into point B. Note that the radius of the arc must be same as step before. See figure 1.12 as the result.

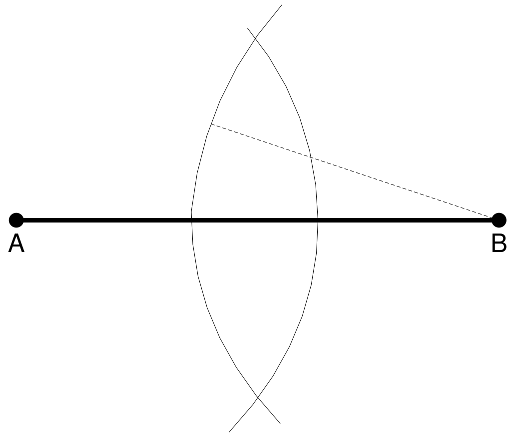


Figure 1.12. Step two

We get there are two intersection between two arcs, let it be M_1 and M_2. Please connect it and we get M_1M_2 as perpendicular bisector of segment AB. See figure 1.13 as the result.

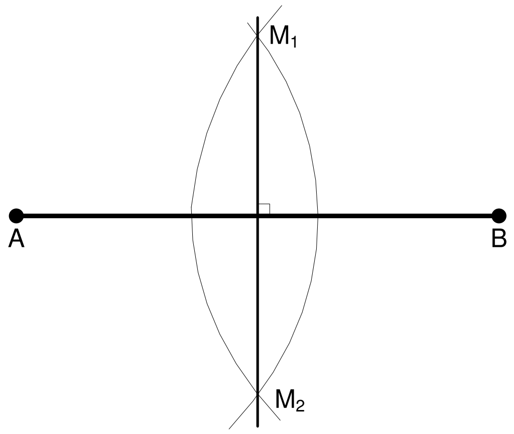


Figure 1.13. Step three

Now, we have got an intersection point of segment AB and M_1M_2. Please give it label by alphabet O. And then O is the mid point of AB. See figure 1.14 as the result.

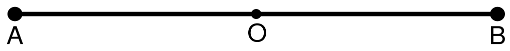


Figure 1.14. Point O as the midpoint of segment AB

That's all the construction of dividing a line segment into two similar parts.

2. **Construction 2: Drawing a line by a point outside of a line (not linear) and perpendicular to the given line.**

A little different from the construction one, in construction two we draw the segment first. In construction one, the segment AB is given but in second construction we build the segment.

Question!
Draw a line which is perpendicular to AB and pass through the point P.

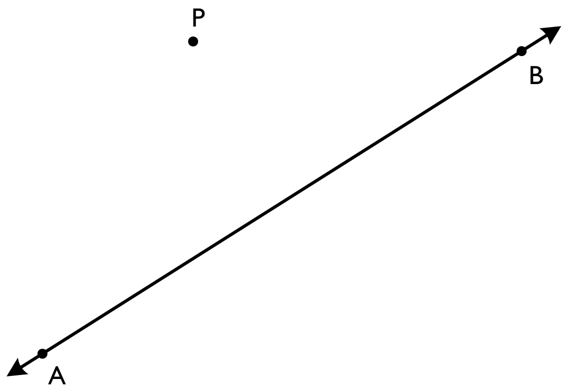


Figure 1.15. Point P outside of line AB

Answer!
The principal concept to solve the problem is making perpendicular bisector through point P. The problem now is, which vertex on line that we will choose as center of the arc. From point P, make an arc with P as the center. Choose radius as yours such that the arc intersect the line in two points, namely K and L. *See figure 1.16.*

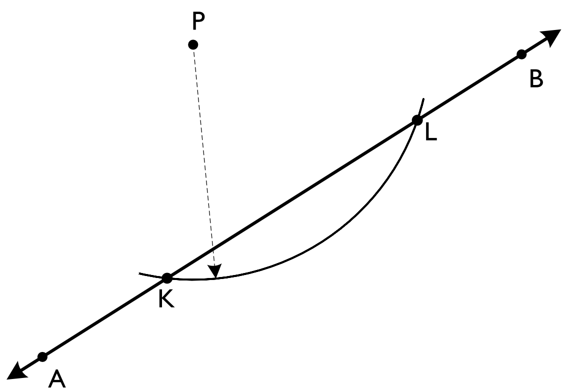


Figure 1.16. Point P center of arc KL. The dashes are made from compass.

As the center, make an arc from K and L, which has radius more than a half of $|KL|$. The intersection between arc K and arc L is given name Q. Now, we have explained that the points like construction 1 are K and L, and we build that.

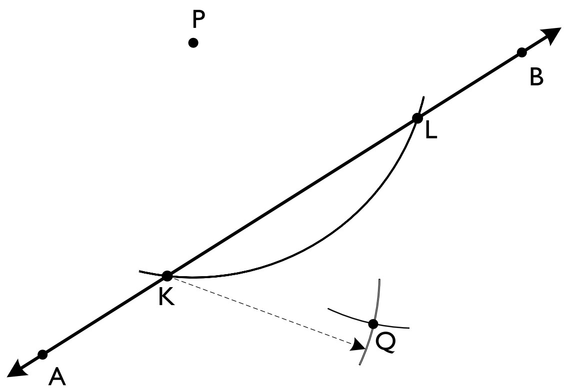


Figure 1.17. Point Q is the common vertex of arc K and arc L..

Make a line that is connecting P and Q and we get a line PQ that is perpendicular to line AB through point P outside the line of AB.

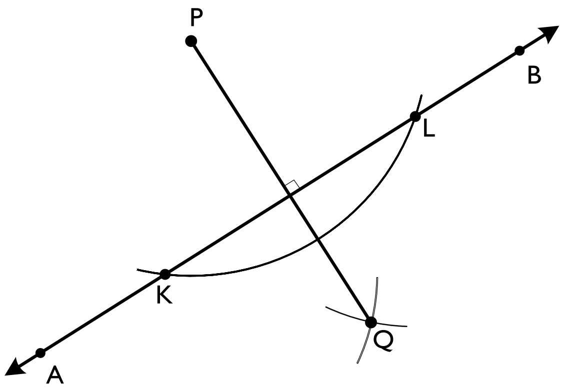


Figure 1.18. Line PQ $\perp$ AB, pass through point P outside the line

3. Construction 3: Drawing a perpendicular line from a point on a line.

Sometimes we were asked to build a perpendicular line from a point lie on the line given. The principal is also like to problems before. Our duty is just find which is the vertex be the center of arc.

Question!
Let line AB and point P $\in$ AB. Draw a line m such that m $\perp$ AB and m pass through point P.

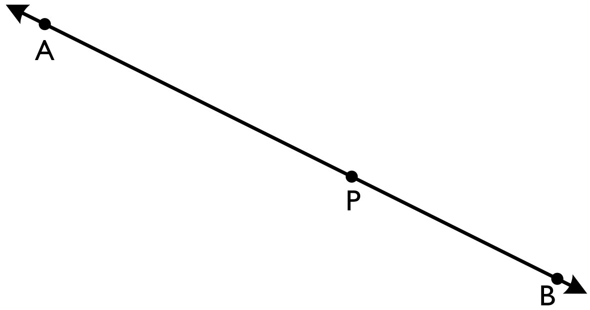


Figure 1.19. Point P lie on line AB

Answer!
We do step like this. With P as the center, make two arcs intersect the line AB. Give name with K and L.

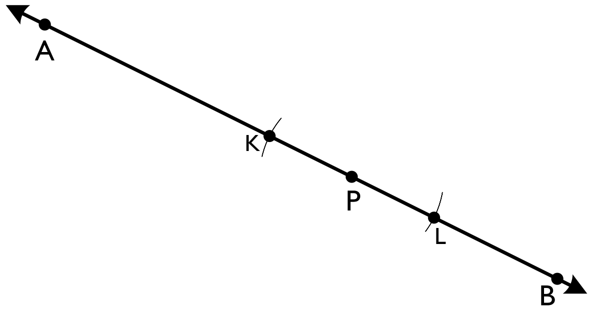


Figure 1.20. 2 intersection point K and L from P as center of arc

Next step is like this. As the center, make an arc from K and L, which has radius more than a half of $|KL|$. The intersection between arc K and arc L is given name Q. See figure 1.21 as the result.

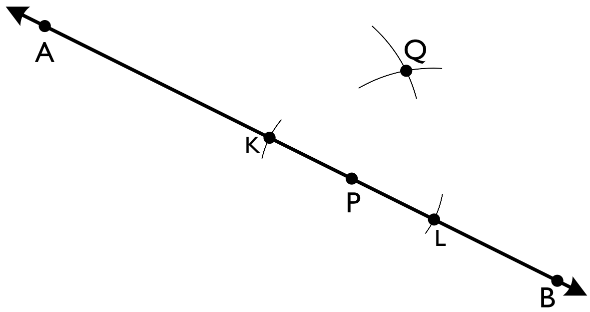


Figure 1.21. Finding the point Q

The last step is just connecting P and Q. We get PQ $\perp$ AB. See figure 1.22 as the result.

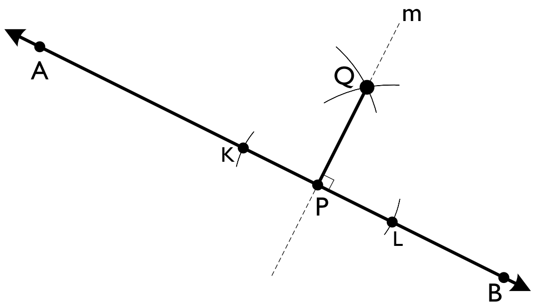


Figure 1.22. Line m ⊥ AB.

That's all the construction of drawing a perpendicular line from a point on a line.

4. Construction 4: Dividing an angle into two similar size angles (making angle bisector).

Angle bisector is one important concept of geometry. We will often use this concept to proof or explain other related topics. To understanding the concept, we will give question and the answer is the step of construction 4.

Question!
Make a line pass through the vertex A such that ∠A is divided into two similar size angles.

Answer
Let ∠A as shown on figure 1.23.

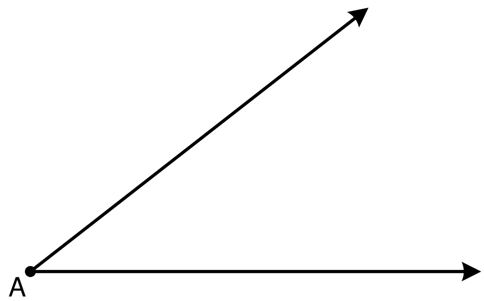


Figure 1.23. ∠A given

With point A as center, draw an arc such that the arc intersects both of angle legs. The radius of the arc is up to you. Let give name the intersection with B and C. See figure 1.24 as the result.

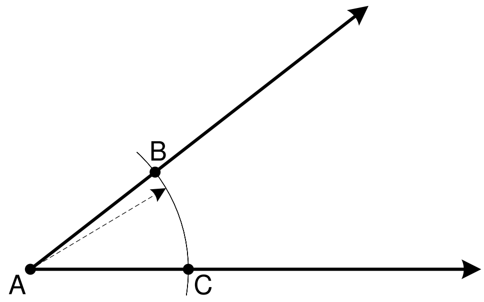


Figure 1.24. Point B and C as result of intersection

From B and C draw a similar length arc, such that both of arcs have intersection inside the angle and in front of ∠A. Let give the intersection name as P, see figure below as the result.

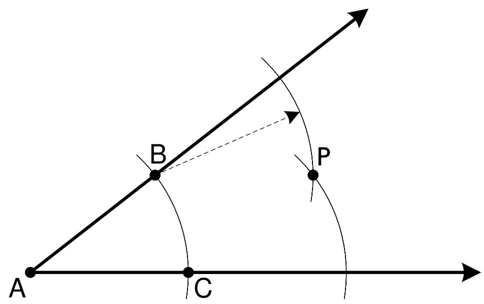


Figure 1.25. Point P as result of intersection between arc B and arc C

The last step is just connecting A and P, and we get m∠BAP = m∠CAP.

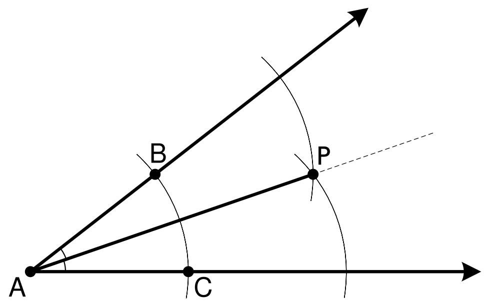


Figure 1.26. We get m∠BAP = m∠CAP. AP is angle bisector

That is the construction of dividing an angle into two similar size angles (making angle bisector). Because of AP is an angle bisector, we may say it AP bisects ∠A.

5. Construction 5: Moving (or duplicating) an angle

This concept is often used in triangle concept. We usually give some angle and ask the reader to make triangle by the angles given. To understanding the concept, we will give question and the answer is the step of construction 5.

Question!
It is given angle A below. Draw the duplicate of angle A to line g, which is M as the angle point in line g.

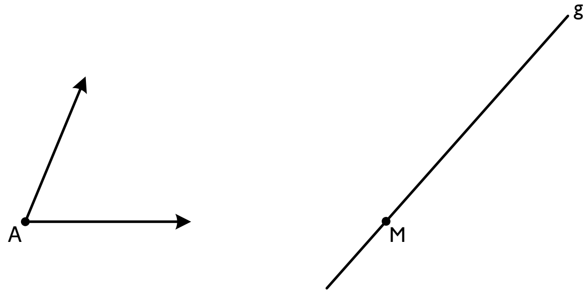


Figure 1.27. Duplicate ∠A to line g

Answers!

With point A as the center, make an arc until intersect both of its legs. Let give the intersection name as B and C. See figure 1.28 as the result.

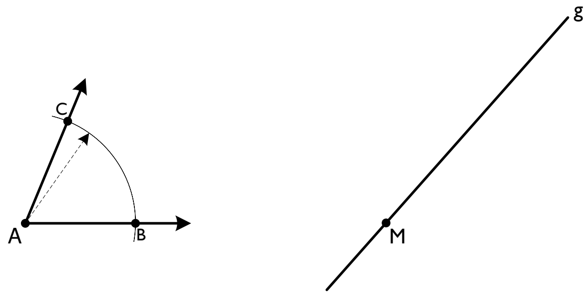


Figure 1.28.

Do it with similar radius length on the line g, such that we get MF has similar length to AB.

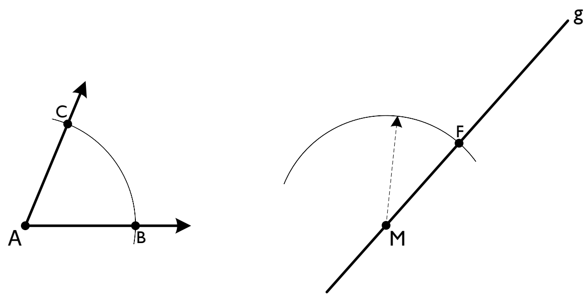


Figure 1.29.

From B, draw an arc with radius length is $|BC|$. Do similar step has just before with M as the center of arc. We get intersection between the arcs on step before with this one. Let give the intersection name with R. See figure 1.30 as the result.

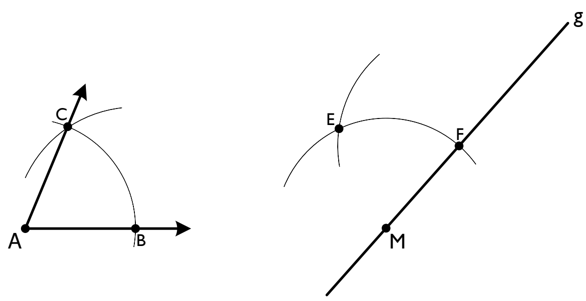


Figure 1.30.

Connect F with E, then we get m∠CAB = m∠EMF.

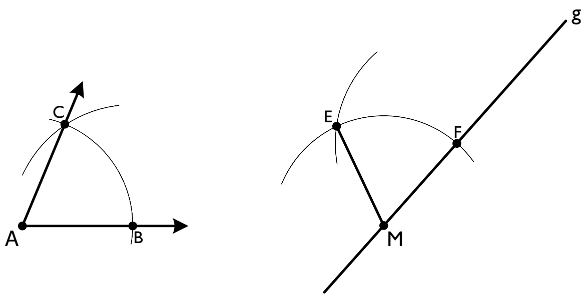


Figure 1.31.

D. EXERCISE Ch. 1

1. Radio station KIDS broadcasts with sufficient strength so that any town 100 kilometers or less but no further from the station's antenna can receive the signal.
   (a) If the towns of Kutha and Metro pick up KIDS, what can you conclude about their distances from the antenna?
   (b) If a messenger were to travel from Kutha to the antenna and then on to Metro, he would have to travel at most how many kilometers?
   (c) What is the maximum possible distance between Kutha and Metro? Explain why your answer is correct.

2. Charts indicate that city B is 265 km northwest of city A, and city C is 286 km southwest of city B. What can you conclude about the distance from city A directly to city C?

3. Which of the following sets of lengths could be the lengths of the sides of a triangle:
   (a) 2 cm, 2 cm, 2 cm
   (b) 3 km, 3 km, 2 km
   (c) 3 m, 4 m, 5 m
   (d) I! m, 5 m, 3! m
   (e) 5 cm, 8 cm, 2 cm
   (f) $2\frac{1}{2}$ cm, $3\frac{1}{2}$ cm, $4\frac{1}{2}$ cm

4. If two sides of a triangle are 12 cm and 20 cm, the third side must be larger than … cm, and smaller than … cm.

5. Let P and Q be distinct points in the plane. If the circle of radius r_1 around P intersects the circle of radius r_2 around Q in two points, what must be true of d(P, Q)?

6. If d(X, Y) = 5, d(X, Z) = $1\frac{1}{2}$, and Z lies on XY, then d(Z, Y) = ?

7. Draw a line segment AB whose length is 15 cm. Locate points on AB whose distances from A are:
   (a) 3 cm; (b) $\frac{5}{2}$cm; (c) $7\frac{1}{2}$ cm; (d) 8 cm; (e) 14 cm

8. Let X and Y be points contained in the disk of radius r around the point P. Explain why $d(X, Y) \leq 2r$. Use the Triangle Inequality.

Chapter 2

THE LOCUS OF GEOMETRY OBJECTS

Locus is basic knowledge to learn geometry. That is locus is not easy to learn about. We assume that in locus of basic geometry we learn about (a) distance, (b) basic parallel law, (c) and perpendicularly. Why basic parallel law? Because there are some topics that couldn't be explained before we learn polygon. So, special for parallel, we divide it into some parts, and we put some material here.

A. DISTANCE

There three ideas about distance that we will discuss. There are:

1. distance between two points,
2. distance between points and line, and
3. distance between two parallel lines.

We define the **distance** between two points P and Q in the plane as the length of the line segment connecting them, which we have already denoted d(P, Q). Keep in mind that this symbol stands for a number. We often write |PQ| instead of d(P, Q).

A few ideas about distance are obvious. The distance between two points is either greater than zero or equal to zero. It is greater than zero if the points are distinct; it is equal to zero only when the two points are in fact the same-in other words when they are not distinct. In addition, the distance from a point P to a point Q is the same as the distance from Q back to P.

Definition.
2.1. We write these properties of distance using proper symbols as follows:
- (a) For any points P, Q, we have d(P, Q) > O. Furthermore, d(P, Q) = 0 if and only if P = Q.
- (b) For any points P, Q, we have d(P, Q) = d(Q, P).

The situation could be shown as figure below:

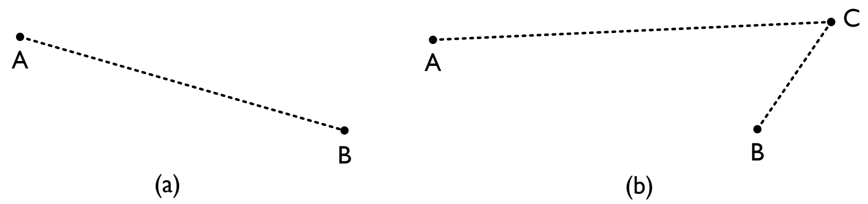


Figure 2.1. Distance between two points

The length of segment between point A and B is the distance between A and B (see figure 2.1 (a)) but why the length segment BC and add by the length of segment CA is not the distance between A and B (see figure 2.1 (b)).

We shall assume that distance between two points is the **length of shortest segment** between two points given.

In the next condition, we will define the distance between a point and a line. We assume that a point has distance to a line if the points is not lie on the line that is call it is not **collinear** or have 0 (zero) distance if the point is lie on the line.

Definition.
2.2. Let any point P and line g. The distance between P and m, denoted by d(P, m) is:
 (a) The length of segment PQ, d(P,Q), $PQ \perp m$, $Q \in m$, if $P \notin m$.
 (b) 0, if $P \in m$.

The situation could be shown as figure below:

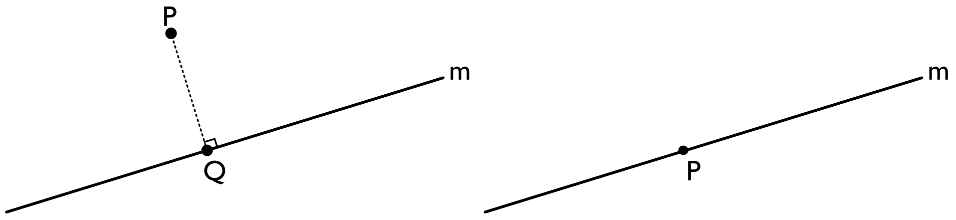


Figure 2.2. Distance between point and line

The written $P \in m$ means the point P is lie on the line m, and vice versa. The contrast theorem about distance between point and line is shown below.

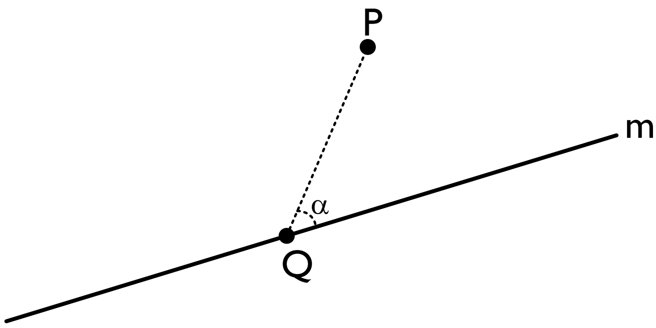


Figure 2.3. The contrast distance principal

The distance between two lines only could hold if two lines are parallel. We assume in planar geometry there only two conditions between two or more lines, that is parallel and intersect each other. No more condition we define in this topic.

Definition.
2.3. Let two parallel line g and h. Let $P \in g$ and $Q \in h$, any point lie on the line.
 The distance between g and h, denoted by d(g, h) is:
 (a) The length of segment PQ, d(P,Q), $PQ \perp g$ and $PQ \perp h$, if $g \neq h$.
 (b) 0, if $g = h$.

The situation could be shown as figure below:

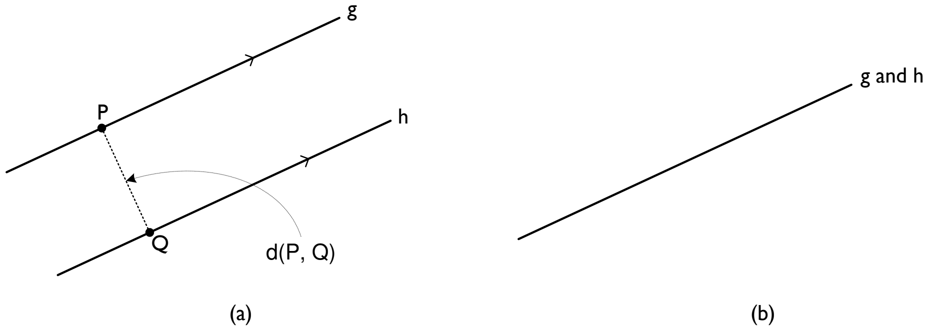


Figure 2.4. Distance between two parallel lines

The written g = h means the line g and h are **coinciding** or both lines **coincide**. The figure 2.4 (a) show us about the condition of distance between two parallel line, and figure 2.4 (b) show us about the condition of coinciding two lines.

B. PARALLEL LAW

In this sub topic we only explain some subject material that is used as based theory. We will continue this topic to higher level of parallel law in chapter 5, the next parallel law. Some proof in chapter 5 needs the science about polygon, so we put the subject material to the next level.

A. Two Lines Intersects by Third Line

We will define some condition of angle and line while three unparallel lines have intersected each other. The situation is shown below.

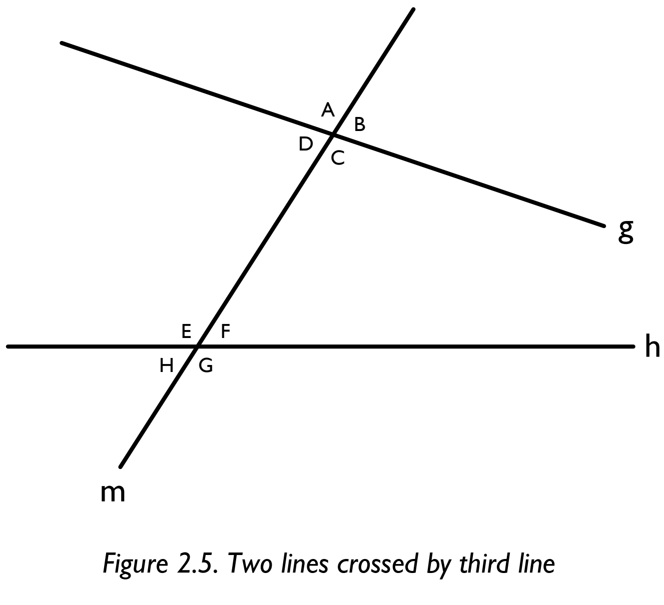


Figure 2.5. Two lines crossed by third line

If two lines g and h intersect by one line m, it will build 8 angles. The condition are:

a. <u>*Corresponding angles,*</u> the condition is hold for $\angle A$ and $\angle E$, $\angle B$ and $\angle F$, $\angle C$ and $\angle G$, $\angle D$ and $\angle H$. If g || h then the corresponding angles will be called by **parallel angle**.

b. *Interior corresponding angles*, the condition is hold for ∠C and ∠F, ∠D and ∠E.
c. *Exterior corresponding angles*, the condition is hold for ∠B and ∠G, ∠A and ∠H.
d. *Interior alternate angle*, the condition is hold for ∠D and ∠F, ∠C and ∠F.
e. *Exterior alternate angle*, the condition is hold for ∠A and ∠G, ∠B and ∠H.

B. Parallel Lines

Our discussion is about the locus of lines on the plane. Our specific topic is about parallel lines. The situation is shown below.

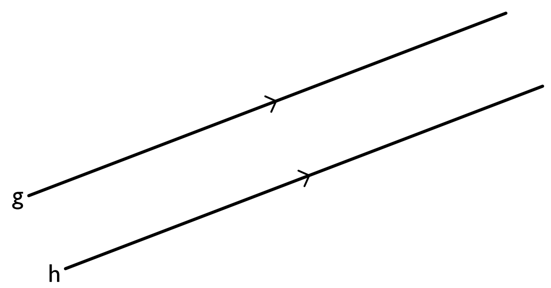


Figure 2.6. Two parallel lines

Definition.
2.4. Two straight line is called parallel if the line lie on same plane and have no common vertex or both of line are not intersecting each other.

Note: if g and h parallel then we denoted it by g || h

Axioms.
2.1. Let two lines g and h. Let there are line m which is intersect line g and h. If the size of corresponding angle is same then g || h.
2.2. If two parallel line intersect by third line, then the **parallel angle** have similar size.
2.3. Let two lines g and h. Let there are line m which is intersect line g and h. If the size of corresponding angles is different then g is not parallel to h, denoted by g ∦ h.

Theorem!
2.1. If two parallel lines are intersected by third line then the interior alternate angles have similar size.

See figure below for more detail explanation.

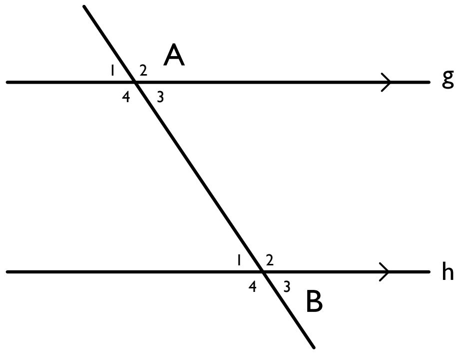


Figure 2.7. Condition of Theorem 2.1

It is given line g and h, intersect by line m.
Prove that $m\angle A_4 = m\angle B_2$.

Proof!
Obvious $m\angle A_4 = m\angle A_2$, because of opposite angle, theorem 1.4.
Obvious $m\angle B_2 = m\angle A_2$, because of corresponding angle, axiom 2.
We conclude that $m\angle A_4 = m\angle B_2$. □

Theorem!

2.2. If two parallel lines intersect by third line then the sum of interior corresponding angles is 180°.

Proof!
Look at figure 2.7.
We will proof that $m\angle A_3 + m\angle B_2 = 180°$.

Obvious $m\angle A_3 + m\angle A_2 = 180°$, because of $\angle A_3$ and $\angle A_2$ are two adjacent angles lie on similar line.
Obvious $m\angle B_2 = m\angle A_2$, because of corresponding angle, axiom 2.
We conclude that $m\angle A_3 + m\angle B_2 = 180°$. □

Theorem!

2.3. If two lines intersect by third line such that the interior alternate angles have similar size then the both of lines are parallel each other.
2.4. If two lines intersect by third line such that the sum of interior corresponding angles is 180° then both of lines are parallel each other.

Proof!
We let the proof as exercise for the reader.

Exercise.

1. If two parallel lines are intersected by third line, prove that:
   a. The size of exterior alternate angles is same,
   b. The sum of exterior correspondent angle is 180°.
2. If two lines are together perpendicular to another line, the both of lines are parallel. Prove it!
3. If two parallel lines are intersected by third line, then the lines which are bisect the correspondent angle are also parallel.

C. PERPENDICULARLY

We define two lines to be **perpendicular** if they intersect, and if the angle between the lines is a right angle. Then this angle has 90°. Perpendicular lines are illustrated below.

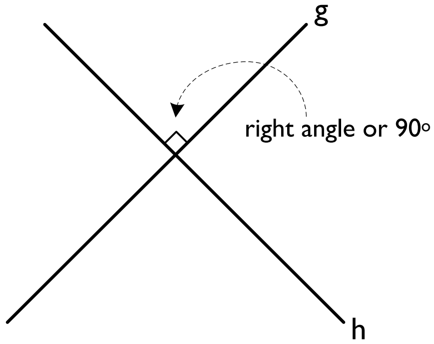


Figure 2.8. Perpendicular

Notice that we use a special " ⏋ " symbol, rather than a small arc, to indicate a 90° (or right) angle. We define two segments or rays to be **perpendicular** if the lines on which they lie are perpendicular.

To denote that lines (or segments) *g and h* are perpendicular, we use the symbols

$$g \perp h$$

As with parallel lines, we assume some properties.

Axioms.

2.4. Given a line g and a point P, there is one and only one line through P, perpendicular to g.

2.5. Given two parallel lines g and h. If a line m is perpendicular to g then it is perpendicular to h.

The condition of axiom 2.4 is shown below.

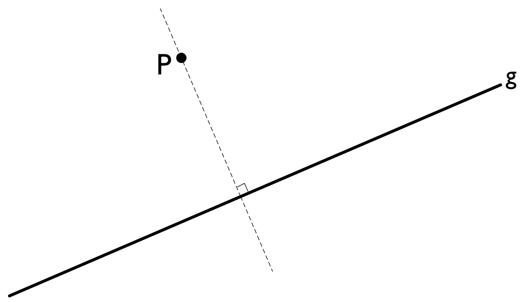


Figure 2.9. Perpendicular line pass through one certain points that is collinear

The condition of axiom 2.5 is shown below.

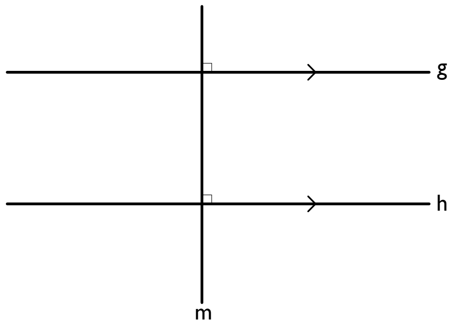


Figure 2.10. Two parallel line intersect by one perpendicular line

Using these two new axioms, we can prove a theorem relating perpendiculars and parallels.

Theorem!

2.5. If m is perpendicular to line g, and m is also perpendicular to line h, then g is parallel to h.

We will try to proof by giving the general counter example. We will proof the contrast is not true to explain that the condition is true.

Proof

Let g, h, m line, $m \perp g$ and $m \perp h$.

Suppose that g and h are not parallel.

Obvious if g and h are not parallel, by definition of parallel line, there is at least a common vertex P, which is $P \in g$ and $P \in h$ too. *See figure 2.11 below.*

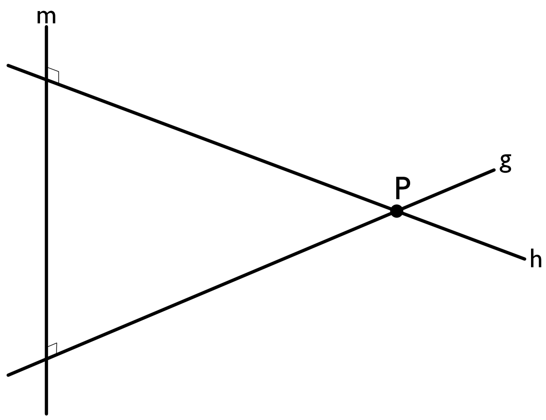


Figure 2.11. The contrast condition

The condition shows us a contradiction between axiom no. 2.4 with it. The axiom tells us there only one line g such that perpendicular to m passes through point P outside the line m.

We get there are a contradiction.
So, the contrast is false.
We conclude that if m is perpendicular to line g, and m is also perpendicular to line h, then g is parallel to h. □

D. EXERCISE Ch. 2

1. Look at figure 2.12 below, line k is perpendicular to line v, and line l is perpendicular to line v. What can you conclude about lines k and l? Why?

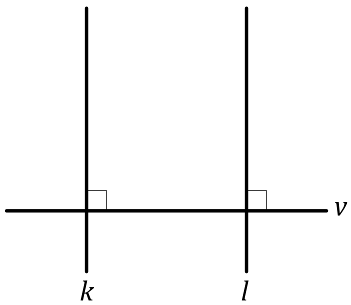


Figure 2.12.

2. In figure 2.13 line m is drawn from point P perpendicular to g. If g and h are parallel, what can you conclude about m and h? Why?

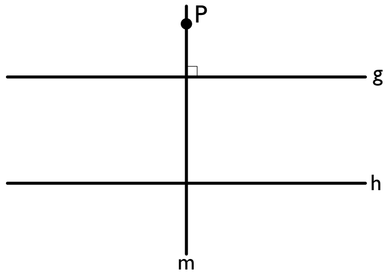


Figure 2.13.

3. In Figure 2.14 below,
   PR is perpendicular to PT and
   PQ is perpendicular to PS.
   Prove that m(∠a) = m(∠b).

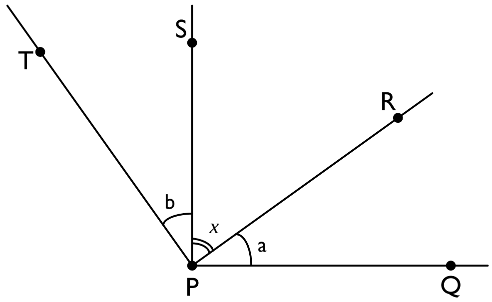


Figure 2.14.

4. Below is an example of a non-mathematical proof by contradiction:

   Johnny wants to go to the store after dinner. His mother says no, he should stay home, do his work, and maintain his straight-A average. Johnny argues: "Suppose I don't go to the store tonight. Then I won't be able to get a protractor. Tomorrow, I'll be without one in geometry class, and the teacher will get mad. As a result, I'll get an F. Since that is an intolerable thing, you must let me go to the store."

   Give another example of a "proof by contradiction" that you might have used some time in your life.

5. Refer to Figure 2.15 below, to answer the following. We assume CD ⊥ AB.
   (a) m(∠l) + m(∠L2) = ___°
   (b) If m(∠3) = 50°, then m(∠4) = ___°
   (c) Is ∠AOT the supplement of ∠TOB?
   (d) M(∠1) + m(∠2) + m(∠3) + m(∠4) = ___°
   (e) If m(∠4) = 23°, then m(∠3) = ___°
   (f) Name, using numbers, two angles that are adjacent to ∠2. ___ and ___
   (g) If m(∠1) = 32°, then m(∠TOB) = ___°
   (h) Must OT be perpendicular to OS if m(∠1) + m(∠4) = 90°?

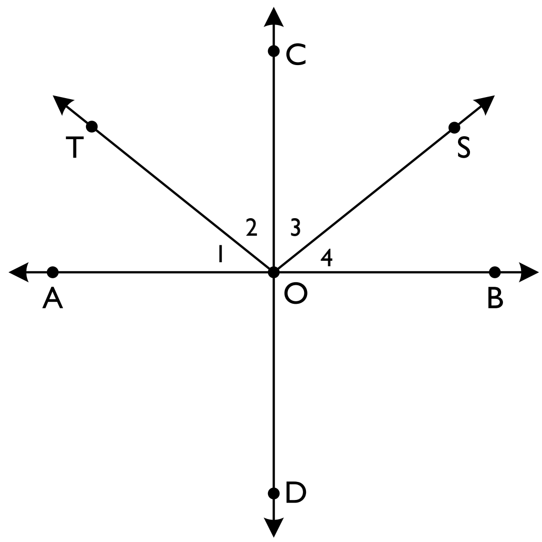


Figure 2.15.

6. Suppose L_1 and L_2 are both perpendicular to L_3, and they both intersect L_3 at point P. What can you conclude about L_1 and L_2?

7. Look at figure 2.16 below. In quadrilateral PBQC, assume that ∠PBQ and ∠PCQ are right angles, and that m(∠x) = m(∠y). Prove that m(∠ABQ) = m(∠DCQ).

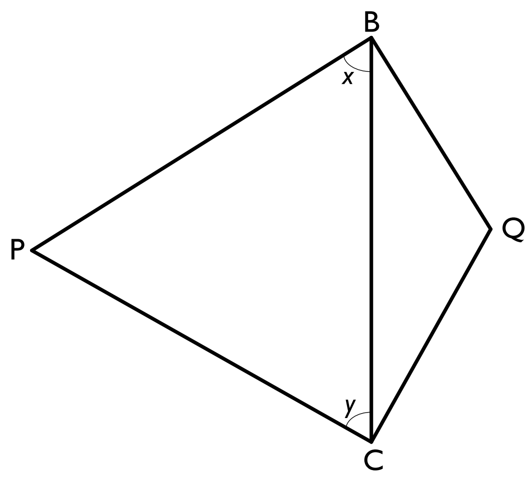


Figure 2.16.

Chapter 3

T R I A N G L E

In this chapter we will discuss about some properties of triangles. For basic definition, we let the reader to arrange as good as reader can do. We let the reader to find their self knowledge and after that, they can arrange the definition of a triangle their self.

A. TRIANGLE AND PART OF A TRIANGLE

The figure realized by connecting three no collinear point by line segments is called a triangle (*Licker. McGraw Hill Co.*). A Triangle ABC sometimes denoted by ∆ABC.

Exercise:
It is given three segment lines. Their lengths are 10 cm, 7 cm, and 8 cm. Draw a triangle with three segments given.

Construction 1:
Given segments line 5, 7, and 10 cm, label it with p, q, r.

1. *Choose one segment given, construct an equilateral triangle, whose sides have the same length as that segment.*
2. *Construct a triangle such that one of the sides has the same length as p, while the two other sides have the same length as q.*
3. *Construct a triangle whose sides have lengths 5 cm, 7 cm, and 15 cm.*
4. *Can you explain why there is trouble with Problem 3?*
5. *Draw any triangle just using a ruler, and measure the length of the three sides. Add up the lengths of any two sides and compare this total with the length of the third side. What do you notice?*
6. *Construct a triangle with sides of length 5 cm, 10 cm, and 15 cm. What happens?*
7. *Let P, Q, and M be three points in the plane, and suppose*

$$d(P, Q) + d(Q, M) = d(P, M).$$

What can you conclude about point P, Q, and M? Draw a picture.

When we learn a triangle, we should understand the parts of a triangle first. Now we define **angle of a triangle** is angle that adjacent to legs of a triangle. And, we define **leg of a triangle** is a segment that is bounded by two different vertices which become angle point of a triangle.

We have known that a triangle is held from three collinear points. The points sometimes we call it **vertices of a triangle**.

Now, we let the figure below to inform the reader more knowledge.

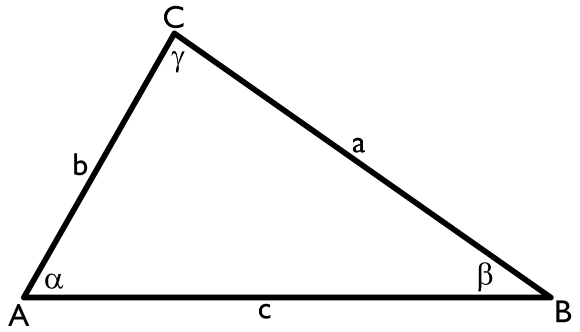


Figure 3.1 Parts of a triangle

The parts of a triangle are:
1) Angle of triangle : ∠A, ∠B, and ∠C;
2) Legs of triangle : AB, BC, AC;
3) Side BC on the opposite of ∠A is labeled by a,
side AC on the opposite of ∠B is labeled by b,
side AB on the opposite of ∠C is labeled by c;
4) ∠A is also called angle α, ∠B is also called angle β, ∠C is also called angle γ.

Before continue to next subject material, let us show some contrast about the definition of a triangle. See figure below.

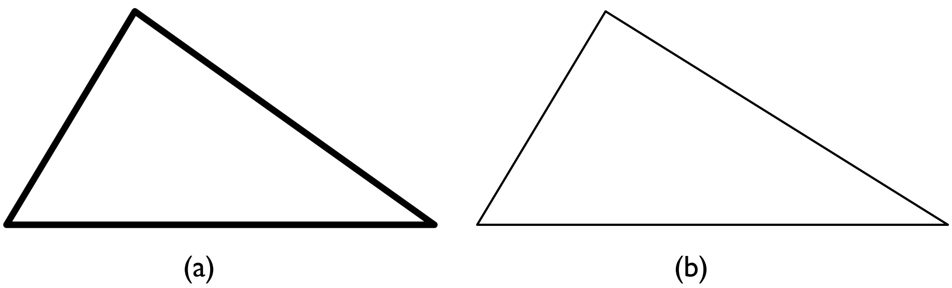


Figure 3.2 Contrast of a triangles

We assume that the planar on figure 3.2 (a) is made from wire or straw. It means, the region surrounded is empty. Thus, the planar on figure 3.2.(b) is assumed made from paper or carton.

My question is, from the definition above, which figure ((a) or (b)) is good explaining the condition of a triangle? And why?

Please compare your knowledge to figure 3.3 below.

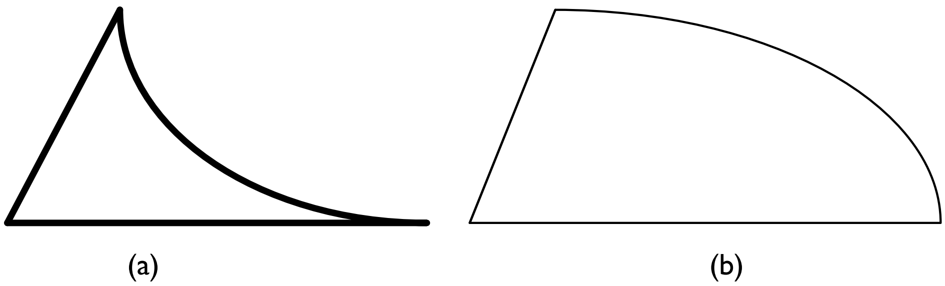


Figure 3.3 Contrast of a triangles

Let we see again the information about triangle below.

**) The figure realized by connecting three no collinear point by line segments is called a triangle (Licker. McGraw Hill Co.).*

From the information above we conclude that figure 3.2 (a) is a triangle. So, it consequence must be figure 3.2 (b) is not a triangle. Yes, sometimes we call figure 3.2 (b) as **the region of a triangle**. Based on the information, we assume that a triangle have no area. The planar which has area sometimes we call it the region, in this case figure 3.2 (b), the region of a triangle, could be mentioned the area.

And, what about the perimeter? We assume that both of the figure (3.2 (a) and (b)) has it perimeter, because we can find the length of segment around the region inside, although the middle region could be empty.

What about figure 3.3, is it a triangle? Off course not. Even figure 3.3 (a) is also made from wire, we can't say it a triangle. We assume that segment in *) means a part of straight line (*remember on chap. 1 pre geometry*). We know that one side on figure 3.3 is not a segment. So we can say that both of figures on 3.3 are not a triangle. That's all basic information about a triangle.

B. CLASSIFICATION OF TRIANGLE BASED ON ITS SIDE AND ANGLE

1) Based on its side, triangles is classified into:

a) Isosceles triangle.

An isosceles triangle is a triangle with two sides equal in length. The two similar sides are called legs of isosceles triangle. The third side is called base of triangle. In a triangle, the angle opposite the base is called the vertex. Both of the other angles is called base angle.

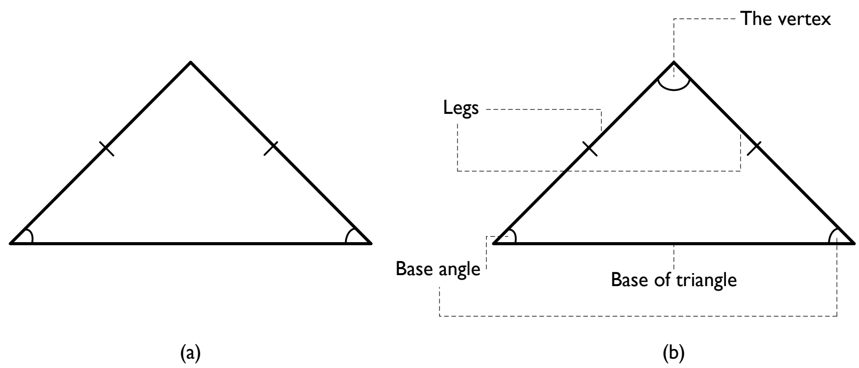


Figure 3.4 Parts of an isosceles triangle

Figure 3.2 (a) shows us the basic form of an isosceles triangle. And the second figure shows us parts of an isosceles triangle.

b) Equilateral triangle.

An equilateral triangle is a triangle which has three equal sides. Because of the side has similar lengths then the size of each angle is also similar, that is 60°.

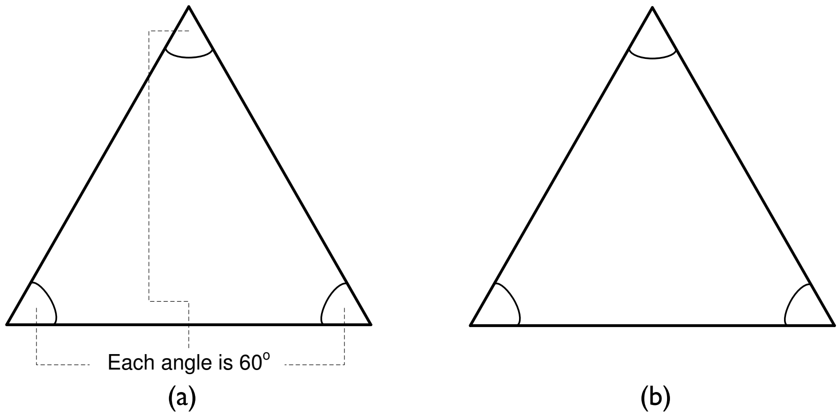


Figure 3.5 Parts of an equilateral triangle

Figure 3.3 show us the angles of 60°. It means, the sum of all angles in a triangle is 180°. *(Prove it!)*

c) Scalene triangle.

A scalene triangle is a triangle which each side has different length.

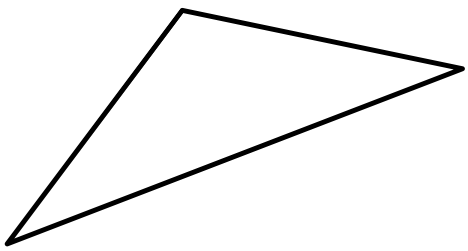


Figure 3.6 Parts of an equilateral triangle

In scalene triangle, and also equilateral triangle, we can say that all side of the triangle is base of triangle. It is different from isosceles triangle; the base of an isosceles triangle is specific side. The base of an isosceles is a side which has different length from another two sides.

2) **Based on its angle**, triangles are classified into:

a) Acute triangle. A triangle is called an acute triangle if all three its angles are acute angle.
b) Obtuse triangle. A triangle is called obtuse triangle if there is one angle in the triangle that is obtuse angle.
c) Right triangle. A triangle is called right triangle if there is one angle in the triangle that is right angle.

The figure of triangles based on information above is shown below.

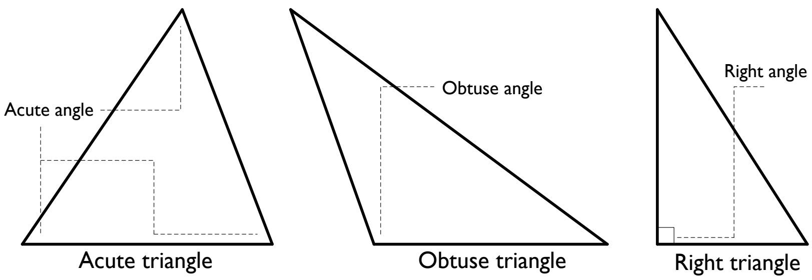


Figure 3.7 Parts of an equilateral triangle

Different from another two kinds, its only acute triangle that has three acute angles. The other kinds of triangle only have one angle which is corresponding.

C. THE SUM OF ANGLES ON A TRIANGLE AND EXTERIOR ANGLE

Theorem!
3.1. The sum of all angles in a triangle is 180°.

Proof!
We leave the proof as the exercise for the reader.

Definition
3.1 Exterior angle is an angle which is adjacent to one angle in the triangle given. The angle and its exterior angle make a straight line.

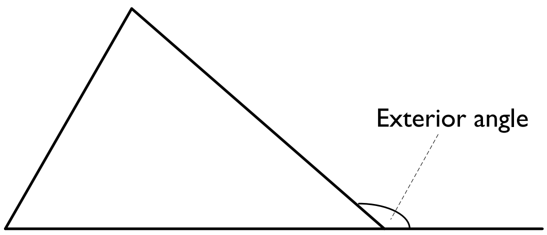


Figure 3.8 An exterior angle

Figure 3.8 show us the exterior angle. We assume that an exterior angle is a supplementary angle of an angle on triangle.

Theorem!
3.2. The size of exterior angle equals to the sum of both other angle.

Analyze.
Look at the figure below!

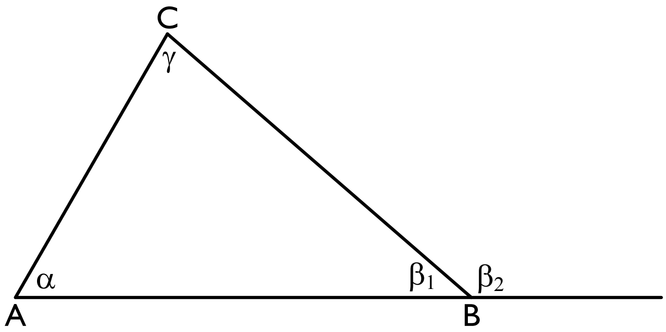


Figure 3.9 Condition of th. 3.2

Proof!
We will proof that m∠β_2 = m∠α + m∠γ.
Obvious m∠β_1+ m∠β_2 = 180°. (*Because of straight line*)
Obvious m∠α + m∠β_1 + m∠γ = 180°. (*Tr. 3.1*)
We get m∠β_1 + m∠β_2 = m∠α + m∠β_1 + m∠γ
⇔ m∠β_2 = m∠α + m∠γ.
So in a triangle the size of exterior angle equals to the sum of both other angle.

Theorem!
3.3. The sum of all exterior angle of a triangle equals 360°.

Proof!
We leave the proof as the exercise for the reader.

D. SPECIAL LINE ON A TRIANGLE

1) Median (median of a triangle)

Definition!
3.2 Median is a line that is connecting a vertex to midpoint of opposite side.
In other sentences, a bisector is a locus of equidistant vertices from its legs.

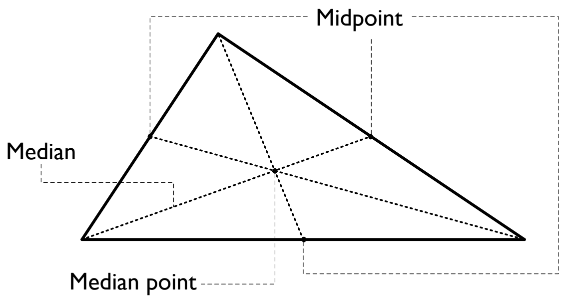


Figure 3.10 A median of a triangle

Three medians were through one certain point. The point is called **median point**. Median point dividing a median by the ratio 2 : 1. For next level, we know that a median divide a triangle into to similar part which has same area. See figure below.

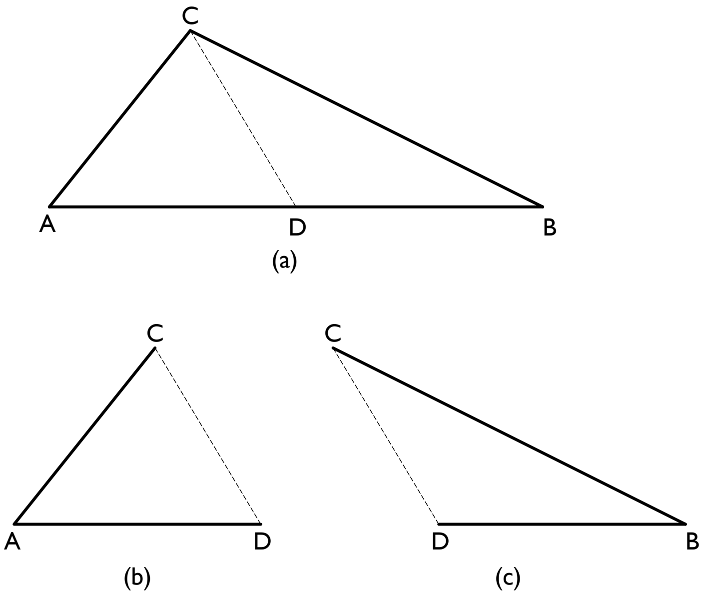


Figure 3.11 A median of a triangle divide

Triangle in figure 3.11 (a) divided by median CD into two parts, those are 3.11 (b) and 3.11 (c). We know that |AD| = |DB|, and **the altitude** (*we will learn soon*) of ΔCAD = ΔDCB. In our discussion later, we can conclude that the area of ΔCAD equals to the area of ΔDCB.

2) Bisector (Angle bisector)

Definition!
3.3 A bisector is a line which is dividing an angle into two similar sizes.

See figure 3.12 below, for more detail information.

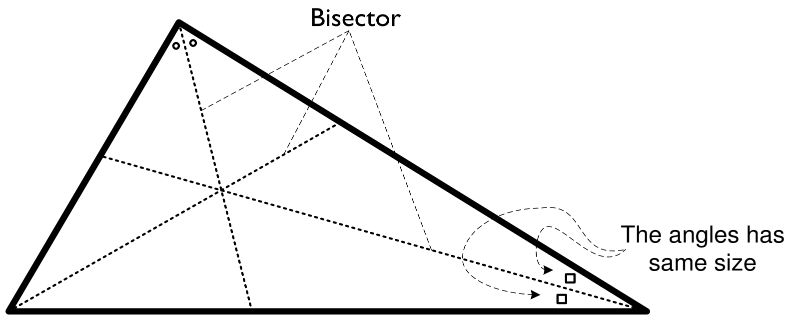


Figure 3.12 A bisector or angle bisector of a triangle

It can be say; the angle bisector was bisecting an angle into two similar size.

Three of bisector has an intersection that is called bisect point. Bisect point is center of inscribed circle.

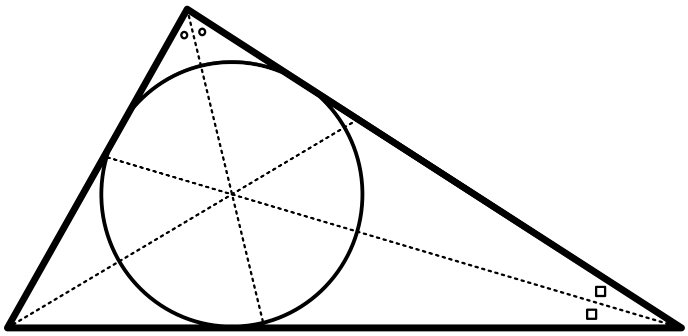


Figure 3.13 Bisector point as center of inscribed circle

3) The altitude

> **Definition**!
> 3.4 The altitude is a line from a vertex perpendicular to opposite side.

To help you understand the condition, see figure 3.14 below.

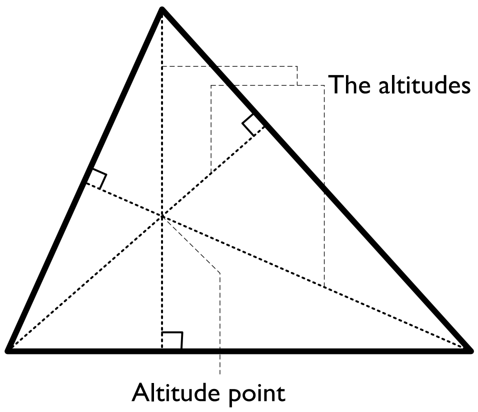


Figure 3.14 Altitudes and an altitude point of a triangle

Three altitudes intersect at one point which is called altitude point.

4) Perpendicular bisector (Axis line)

> **Definition**!
> 3.5 Perpendicular bisector of a line or side is a line that is perpendicular to the side or line through the midpoint.

To help you understand the condition, see figure 3.15 below.

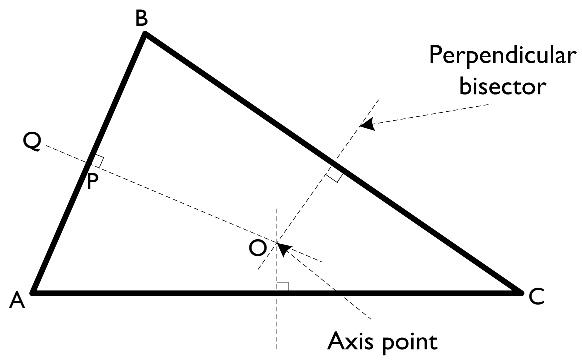


Figure 3.15 Perpendicular bisector and axis point of a triangle

In a triangle perpendicular bisector sometimes called by axis line. All of perpendicular bisector through one certain point that is called axis point. The axis point is the center of **circumscribed** of a triangle.

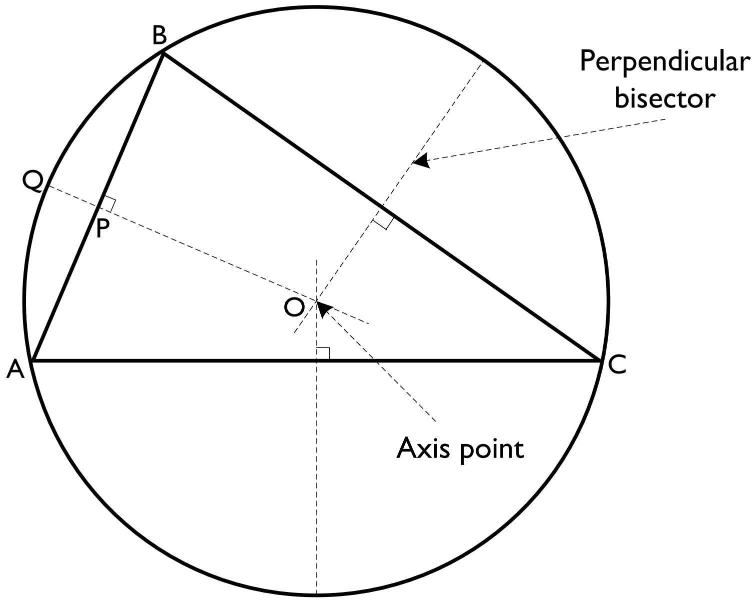


Figure 3.16 Axis point as center of circumscribed of a triangle

In the figure 3.16, we give some note that (i) $OQ \perp AB$, $|AP| = |PB|$. Off course OQ is a perpendicular bisector. It means note (i) is suitable for all perpendicular bisector of a triangle.

We have explained all four special lines of a triangle. We will give some properties about the special lines below.

Theorem!
3.4. Inside bisector and outside bisector to similar angle is perpendicular one another.

Analyze.
Look at the figure below.

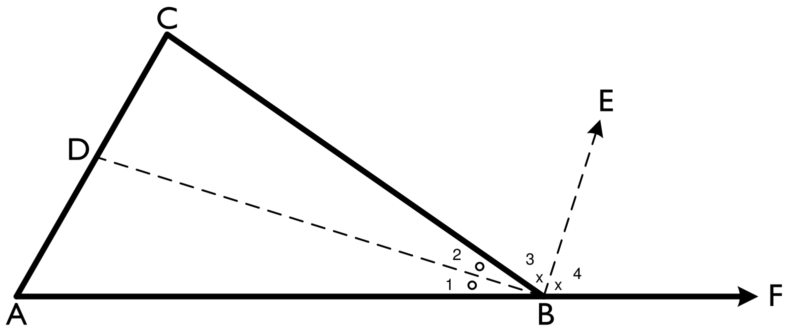


Figure 3.17 Condition of theorem 3.4

Proof!
It is given ΔABC, BD is interior bisector, and BE is exterior bisector.
We will proof that BE ⊥ BD.
Obvious $m\angle B_1 = m\angle B_2$ and $m\angle B_3 = m\angle B_4$
Obvious $m\angle B_1 + m\angle B_2 + m\angle B_3 + m\angle B_4 = 180°$
$\Leftrightarrow 2 \,.\, m\angle B_2 + 2 \,.\, m\angle B_3 = 180°$
$\Leftrightarrow 2 \,.\, (m\angle B_2 + m\angle B_3) = 180°$
$\Leftrightarrow m\angle B_2 + m\angle B_3 = 90°$.
Because of $m\angle B_2 + m\angle B_3 = 90°$ then we conclude that BD ⊥ BE.

E. CONGRUENCY

We shall say that two figures in the plane are similar whenever one is congruent to dilate of the other. Therefore the two quadrilaterals are similar, since one is just an enlargement of the other. Any two circles are similar, if the two circles have the same radius. We simply take dilation by 1 to satisfy the definition. For the moment, we will study similar triangles, as illustrated below.

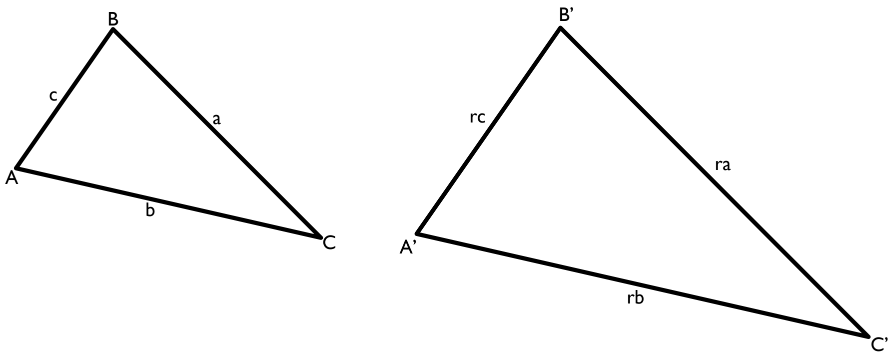


Figure 3.18 Dilation of a triangle

We can easily generate similar triangles by dilating a triangle with respect to one of its vertices or with respect to a point 0 not a vertex like shown below.

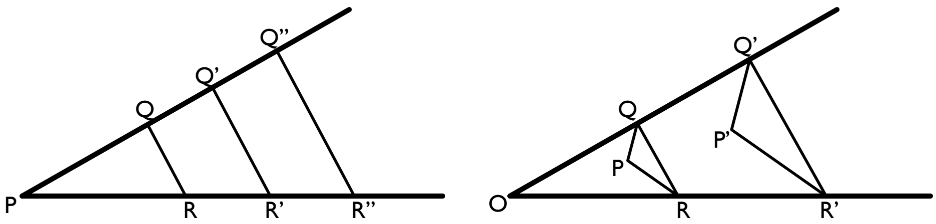


Figure 3.19 Process of dilating a triangle

Let T be a triangle whose sides have lengths a, b, c respectively. If we dilate T by a factor of r, we obtain a triangle which we denote by rT. The lengths of its sides will be ra, rb, rc, as we saw in the preceding section. Note that r can be any positive number. For instance in Figure below we have drawn triangles T, $\frac{1}{2}$T, and 2T.

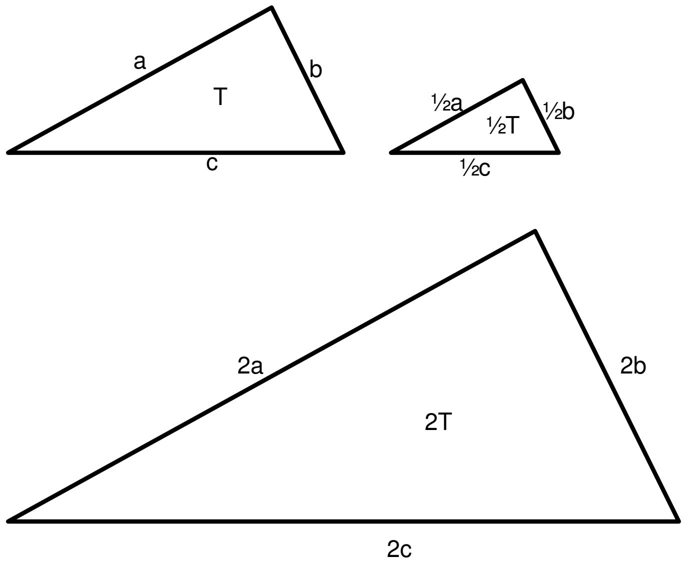


Figure 3.20 Sample of dilating process of a triangle

Denote by T' the dilation of T by r. Let a', b', c' be the lengths of the corresponding sides. Then we have

a' = ra, b' = rb, c' = rc.

Therefore the ratios of the corresponding sides are all equal, that is:

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = r$$

We have seen that if two triangles are similar, then the ratios of the lengths of corresponding sides are equal to a constant r. We now prove the converse.

Definition!

3.6 Two triangles which it’s all corresponding side have same length are similar and congruent.

Note:
If both of the triangles have same length side, then it is denoted by SSS. Notation for congruent is ≅. Another ways to explain the congruency will be given soon.

Theorem!

3.5. Let T, T' be triangles. Let a, b, c be the lengths of the sides of T, and let a', b', c' be the lengths of the sides of T'. If there exists a positive number r such that

$$a' = ra,\ b' = rb,\ c' = rc,$$

then the triangles are similar.

Proof!

The dilation by $\frac{1}{r}$ of T' transforms T' into a triangle T" whose sides have lengths a, b, c, because

$$\frac{1}{r}.ra = a, \quad \frac{1}{r}.rb = b, \quad \frac{1}{r}.rc = c$$

Therefore T, T" have corresponding sides of the same length. By condition SSS we conclude that T and T" are congruent. Therefore T is congruent to a dilation of T', and triangles T and T' are similar. □

Theorem!

3.6. If two triangles are similar, then their corresponding angles have the same measure.

3.7. If the corresponding angles of two triangles have same measure, then the triangles are similar.

Proof!

Let T, T' be the triangles. Let A, B, C be the angles of T and let A', B', C' be the corresponding angles of T'. Let a, b, c and a', b', c' be the lengths of corresponding sides. Let

$$r = \frac{a'}{a}$$

be the ratio of the lengths of one pair of corresponding sides. Then a' = ra. Dilation by r transforms T into a triangle T" whose sides have lengths

$$a'' = ra,\ b'' = rb,\ c'' = rc$$

respectively. The triangles T' and T" have one corresponding side having the same length, namely

$$a'' = a' = ra.$$

The situation is shown below.

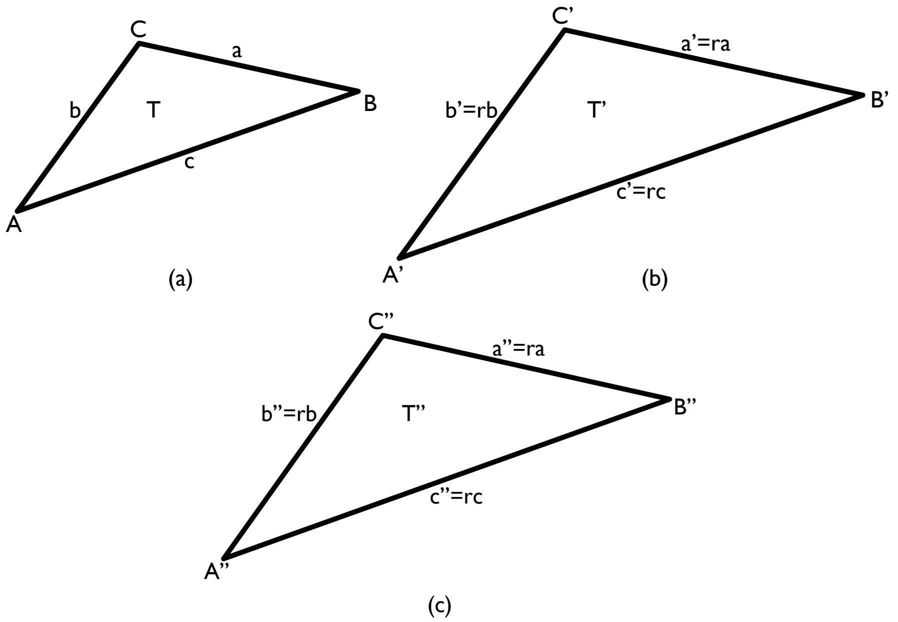


Figure 3.21 Condition of theorem 3.6 and 3.7

We have seen in the previous theorem that dilation preserves the measures of angles. Hence the angles adjacent to this side in T' and T" have the same measure, that is:

$$m(\angle B') = m(\angle B'') \text{ and } m(\angle C') = m(\angle C'').$$

It follows from the **ASA** property that T', T" are congruent. Hence T' is congruent to a dilation of T, and hence T' is similar to T, as was to be shown. □

Theorem!

3.8. Two triangle are congruent if
   a. there is similar side
   b. angle in that side and angle which is opposite to the side is similar
   then we called it congruent by **SAA.**

3.9. Two triangles are congruent if both of triangles are right angle triangle and one leg and also its hypotenuse is same length.

Proof!

We let the proof as exercise.

NOTE!

If two triangle are congruent, then:

a. The length of corresponding sides are same

b. The magnitude of corresponding angle are same.

Corresponding side is the sides in front of the similar measure angles. Therefore corresponding angle is the angle the angles which is correspond to similar length side.

Now, we will give some examples to make the reader more understand about the concept.

Example #1:

Prove that in the isosceles triangle, the lengths of median to the similar sides are same.

Solution:
Look at figure 3.22 to see the condition about the theorem.

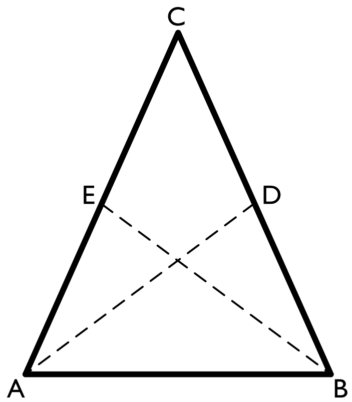


Figure 3.22 Condition to the problem

It is given:
∆ABC is an isosceles triangle with AC = BC.
We build AD and BE be median, and that's why AE = CE and BD = CD.

Prove that AD = BE.

Obvious $BD = \frac{1}{2}.BC.$

Obvious $AE = \frac{1}{2}.AC = \frac{1}{2}.BC$

We get AE = BD.

Look at the ∆ABD and ∆BAE.
We can find fact that:
1. AE = BD
2. m∠BAE = m∠ABD (from given isosceles triangle)
3. AB = AB (coincide)

From fact above and theorems before we conclude that ∆ABD and ∆BAE are congruent, because of **SAS** rule.
So that the length of corresponding side is same include AD and BE.
So, AD = BE. □

We have proof this theorem or maybe have same meaning, but we try to proof by another way, just use the special properties of triangle.

***Theorem**.*
Given line K intersecting parallel lines g and h. Then parallel angles 1 and 5 have the same measure, and so do alternate angles 5 and 3.

Proof.
Look figure below!

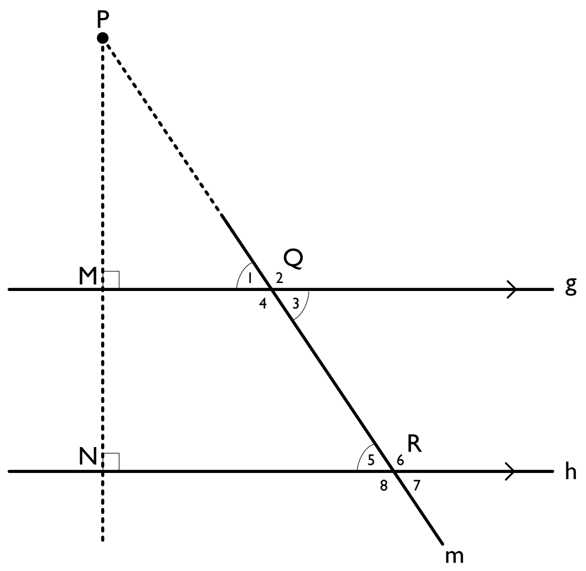


Figure 3.23 Condition to the problem

Through P we draw a line perpendicular to g, intersecting g at M. Since g and h are parallel, AXIOM 2.5 tells us that this new line is also perpendicular to h. We label this point of in tersection N. We have now created two right triangles, Δ.QPM and ΔPRN. By Theorem 3.5, we know that :

$$m\angle R + m\angle P = 90^\circ \text{ (applied to } \Delta PRN)$$
and
$$m\angle Q + m\angle P = 90^\circ \text{ (applied to } \Delta QPM).$$

Therefore $m\angle R_5 = m\angle Q_1$. Since opposite angles have the same measure, we have

$$m\angle Q_1 = m\angle Q_3 \text{ and thus } m\angle R_5 = m\angle Q_3.$$

This proves our theorem. □

F. ANOTHER PROPERTIES ON TRIANGLE

After the explanation above, now, we will explain some properties of a triangle, included its special lines.

Theorem!
3.10. The measure of base angle in an isosceles triangle is same.

See figure below for more detail information.

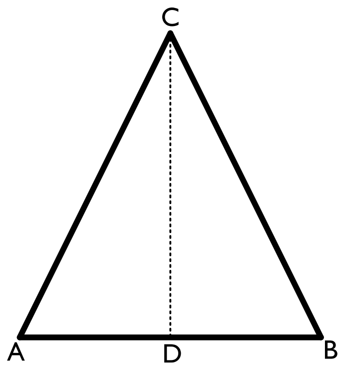


Figure 3.24 Condition of theorem 3.10

It is known:
$\triangle ABC$ is an isosceles triangle, AC = BC. (*see figure 3.24*)
The theorem says that $m\angle A = m\angle B$.

Proof!
Build median CD, and then look at $\triangle ACD$ and $\triangle BCD$.
Obvious (i) AC = BC, (ii) AD = BD, (iii) CD = CD.
So that $\triangle ACD \cong \triangle BCD$
So that $m\angle A = m\angle B$. □

Theorem!

3.11. In the isosceles triangle, the three special lines from the top to the base axis are coinciding.

See figure below for more detail information.

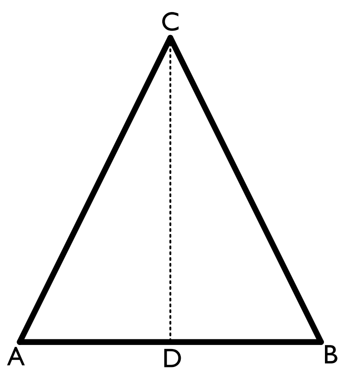


Figure 3.25 Condition of theorem 3.11

It is known:
$\triangle ABC$ is an isosceles triangle, AC = BC, $m\angle A = m\angle B$.
Prove that the median, the altitude, and the bisector of $\triangle ABC$ are coinciding.

Proof!
We build CD as a median of triangle. We will proof that CD is also the altitude and perpendicular bisector.

Look at ΔADC and ΔBDC.
Obvious (i) AC = BC, (ii) m$\angle$ACD = m$\angle$BCD, (iii) CD = CD.
So that ΔACD $\cong$ ΔBCD

The consequences are:
(A) Because of AD = BD, we get CD is a median.
(B) Obvious m$\angle$ADC + m$\angle$BDC = 180°. We get fact that m$\angle$ADC = m$\angle$BDC, so m$\angle$ADC = $\frac{1}{2}$.180° = 90° or CD $\perp$ AB. We get CD is the altitude of ΔABC.
(C) AD = BD and CD $\perp$ AB. We can conclude that CD is perpendicular bisector.

From (A), (B), and (C) we know that the altitude, the median and perpendicular bisector of ΔABC are coincide. □

Theorem!
3.12. If in the triangle, the three special lines from the top to base axis coincide then the triangle given is isosceles triangle.

Proof!
We leave this proof as exercise.

Theorem!
3.13. In the right angle triangle, the length of median to the hypotenuse equals a half of its hypotenuse.

See figure below for more detail information.

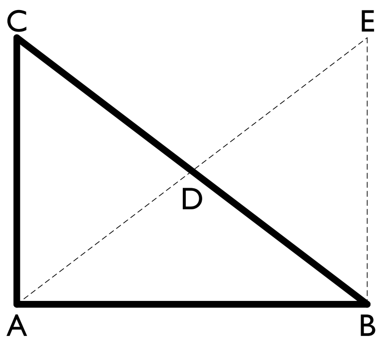


Figure 3.26 Condition of theorem 3.13

It is given
ΔABC, m$\angle$A = 90°.
AD is median of the triangle, such that BD = CD.
The theorem says that $|\mathrm{AD}| = \frac{1}{2}.|\mathrm{BC}|$

Proof!
From the point B, build segment BE parallel to AC until intersect the extension of AD in E. Then ΔADC ≅ ΔBDE, such that AC = BE.

Look at ΔBAC and ΔABE.
We get :
(i) AC = BE
(ii) m∠BAC = m∠ABE = 90°
(iii) BA = AB
From (i), (ii), and (iii) we get ΔBAC ≅ ΔABE.
So that m∠BAE = m∠ABC or ΔABD is isosceles.
We get AD = BD or AD = $\frac{1}{2}$ BC. □

Theorem!
3.14. In the 30° right angle triangle, the length of side opposite to the 30° angle equals a half of its hypotenuse.

See figure below for more detail information.

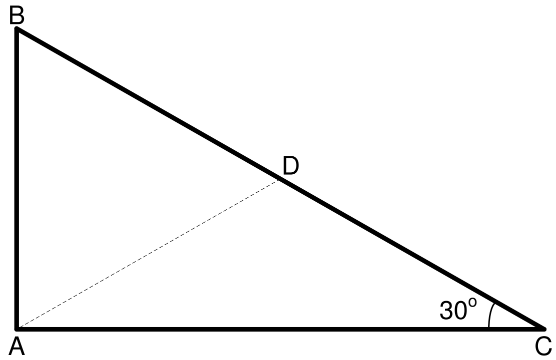


Figure 3.27 Condition of theorem 3.14

Proof!
It is given 30° right angle triangle.

We will show that BC = 2.AB.

Draw a median AD of ΔABC.
Obvious BD = CD.

Look at ΔABD.
Obvious m∠B = 60°
From theorem 1.17 we get AD = $\frac{1}{2}$.BC. And also BD = $\frac{1}{2}$.BC.
We get ΔABD is an isosceles triangle.
The consequence is m∠DAB = m∠ABD = 60°.
Because the sum of angle in a triangle is 180°, we conclude that m∠ADB = 60°.
So that ΔADB is an equilateral triangle.

From the fact above, we get AD = BD = AB = $\frac{1}{2}$.BC.
So we get the length of side opposite to the 30° angle equals a half of its hypotenuse. □

G. EXERCISE Ch. 3

1. Given an isosceles triangle ABC (∠C as top angle). On the base AB, mention a vertex of D and E such that AD = BE.
   Prove that CD = CE !
2. On the triangle ABC, ABC is an isosceles triangle, AB as base of triangle, build a bisector AD and bisector BE. Prove that AD = BE !
3. Draw a perpendicular bisector of a line AB. On the perpendicular bisector, there is a vertex P. Prove that P is equidistant from A and B !
4. Given ΔABC is an isosceles triangle with AB as its base. Bisector AD and BE are intersecting on vertex T. Prove that ΔATE ≅ ΔRTD!
5. Prove that in the isosceles triangle, both of the altitude from base vertices is similar!
6. Given CD as the altitude of ΔABC which is isosceles triangle. Prove that CD the median, the bisector, and also the perpendicular bisector.

H. PROJECTS

It is given some angles and segments below.

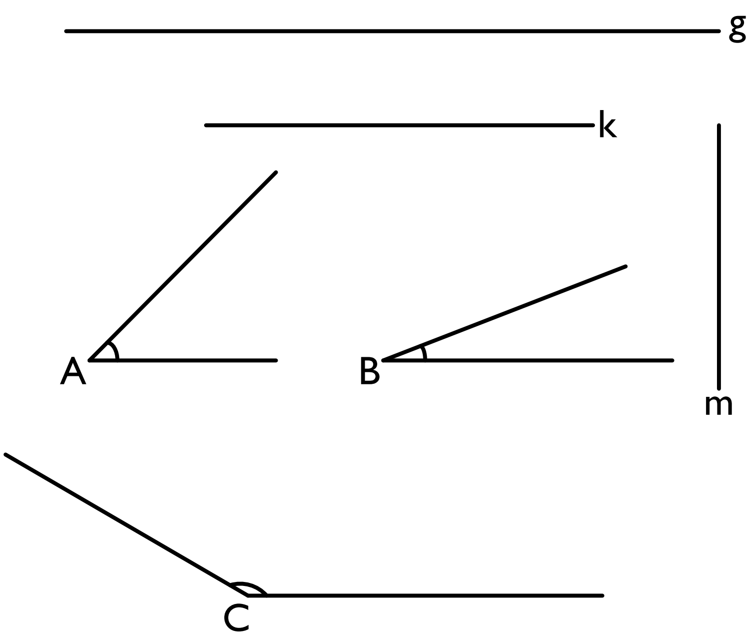


Figure 3.28 Lines and angles given

1. Draw a triangle, if it is known a side and two angle on the side given (g, ∠A and ∠B)
2. Draw a triangle, if it is known two side and one adjacent angle of two side given (g, k, and ∠C)

3. Draw a triangle, if it is known one side, one adjacent angle, and one opposite single to the side given (m, $\angle A$, and $\angle B$)
4. Draw a triangle, if it is known two sides and one opposite angle to any side given (k, m, and $\angle C$)

Chapter 4

P O L Y G O N

In this chapter, we will explain about polygon. We may start from the basic ideas of a polygon, the convexity, basic theorems of polygon, regular polygon, some kinds of polygon, until kinds of quadrilateral. We give you two kinds of exercise in this chapter, because of its large subject material. Some our daily mistakes are also given and directly solved here. We wish for next activity, the mistakes will not be happen again.

A. BASIC IDEAS

Some figure which is polygon is shown below.

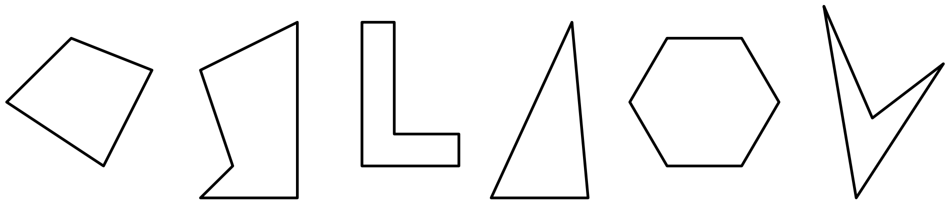


Figure 4.1 Polygon

Some figures which are not polygon are shown below.

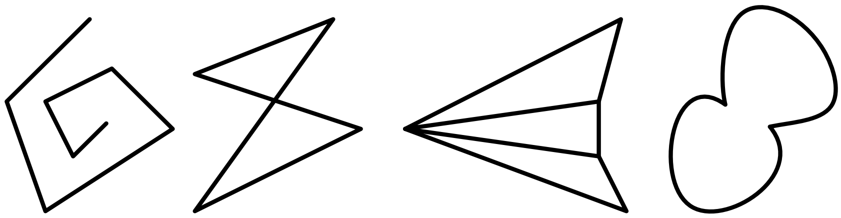


Figure 4.2 Not polygon

We shall give the definition of a polygon in a moment. In these figures, observe that a polygon consists of line segments which enclose a single region.

A four-side polygon is called a **quadrilateral** (Figure 4.1(a) or (f)). A five-side polygon is called a **pentagon** (Figure 4.1(b)), and a six-side polygon is called a **hexagon** (Figure 4.1(e)). If we kept using special prefixes such as quad-, penta-, hexa-, and so on for naming polygons, we would have a hard time talking about figures with many sides without getting very confused. Instead, we call a polygon which has n sides as **n-gon**.

For example, a pentagon could also be called a 5-gon; a hexagon would be called a 6-gon. If we don't want to specify the number of sides, we simply use the word polygon (poly- means many). As we mentioned for triangles (3-gons), there is no good word to use for the region inside a polygon, except "polygonal region", which is a mouthful. So we shall speak of the area of a polygon when we mean the area of the polygonal region, as we did for triangles.

If a segment PQ is the **side** of a polygon, then we call point P or point Q a **vertex** of the polygon. With multisided polygons, we often label the vertices (plural of vertex) P_1, P_2, P_3, etc. for a number of reasons. First, we would run out of letters if the polygon had more than 26 sides. Second, this notation reminds us of the number of sides of the polygon; in the illustration, we see immediately that the figure has 5 sides:

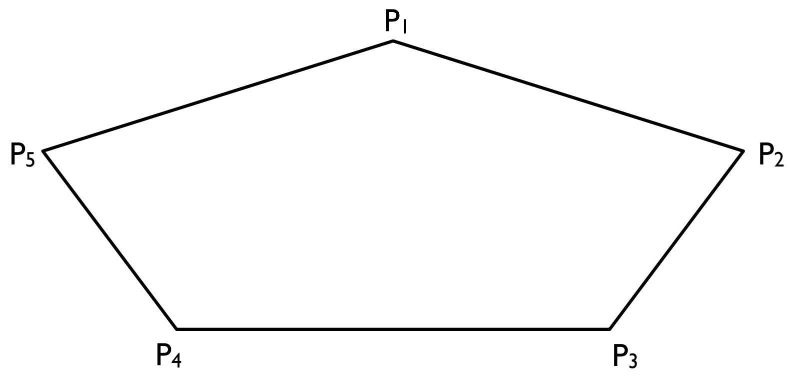


Figure 4.3 A Hexagon

Finally, if we want to talk about the general case, the n-gon, we can label its vertices P_1, P_2 , P_3 , … ,P_{n-1}, P_n as shown:

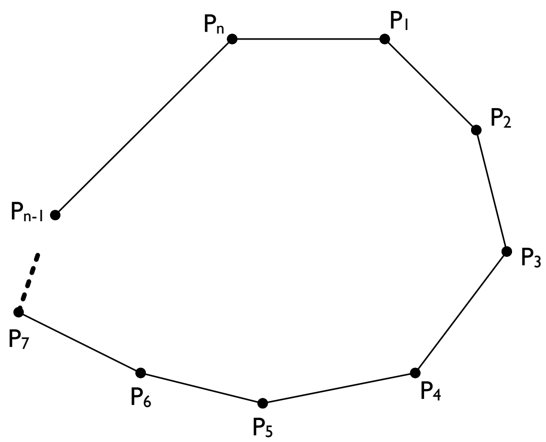


Figure 4.4 General n-gon

We can now define a polygon (or an n-gon) to be an n-sided figure consisting of n segments

$$\overline{P_1P_2},\ \overline{P_2P_3},\ \overline{P_3P_4},\ \cdots,\overline{P_{n-1}P_n},\ \overline{P_nP_1}$$

which intersect only at their endpoints and enclose a single region.

Experiment 4.1.
Below are two rows of polygons. Each polygon in the top row exhibits a common property, while those in the bottom row do not.

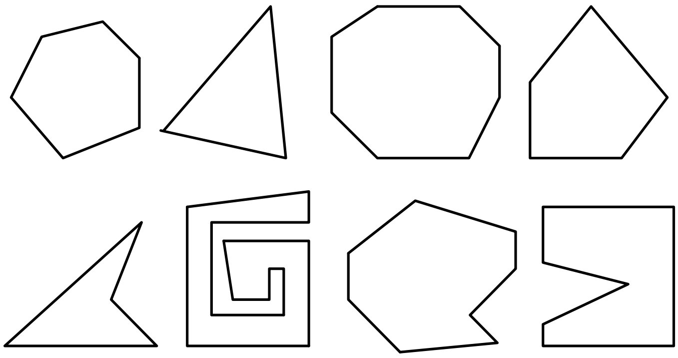


Figure 4.5 Polygons

Can you discover what the top row polygons have in common that the bottom ones do not? Try to state the definition of your property as specifically as possible, using terms and concepts that we have already defined. There are many possible answers.

B. CONVEXITY AND ANGLES

Polygons which look like those in the top row of Figure 4.5 we will call **convex**. Thus we define a polygon to be convex if it has the following property:

> Given two points X and Y on the sides of the polygon, then the segment XY is wholly contained in the polygonal region surrounded by the polygon (including the polygon itself).

Observe how this condition fails in a polygon such as one chosen from the lower row in Figure 4.5:

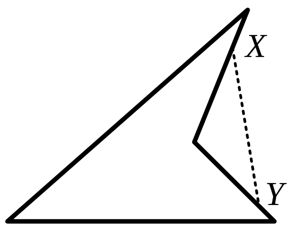


Figure 4.6 Example polygon

You might want to go back to Figure 4.5 and verify that this condition does hold on each polygon in the top row.

Throughout this book we shall only be dealing with convex polygons, as they are generally more interesting. Consequently, to simplify our language, **<u>we shall always assume that a polygon is convex</u>**, and not say so explicitly every time.

In a polygon, let PQ and QM be two sides with common endpoint Q. Then the polygon lies within one of the two angles determined by the rays R_{QP} and R_{QM}. This angle is called one of the **angles of the polygon**. Observe that this angle has less than 180°, as illustrated:

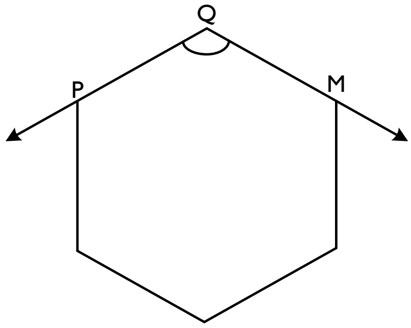


Figure 4.7 Angles of polygon

Experiment 4.2.

1. *Besides the number of sides, two characteristics of polygons are the lengths of its sides and the measures of its angles.*
   a. *What do we call a quadrilateral which has four sides of the same length and which has four angles with the same measure?*
   b. *Can you think of a quadrilateral which has four angles with equal measures but whose sides do not all have the same length? Draw a picture. What do we call such a quadrilateral?*
   c. *Can you draw a quadrilateral which has four sides of equal length, but whose angles do not have the same measure?*
   d. *What do we call a 3-gon which has equal length sides and equal measure angles?*
2. *With a ruler, draw an arbitrary looking convex quadrilateral. Measure each of its four angles, and add these measures. Repeat with two or three other quadrilaterals.*
3. *Repeat the procedure given in Part 2 with a few pentagons, and then a few hexagons.*

What can you conclude? Can you say what the sum of the measures of the angles of a 7-gon would be? How about a 13-gon?
For the rest of this experiment, we will develop a formula to answer these questions.

Consider a convex quadrilateral. A line segment between two opposite vertices is called a **diagonal**. We can decompose the quadrilateral ("break it down") into two triangles by drawing a diagonal, as shown:

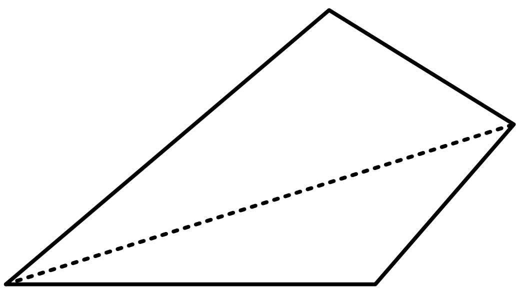


Figure 4.8 A diagonal

Notice that the angles of the two triangles make up the angles of the polygon. What is the sum of the angles in each triangle? In the two triangles added together? And in the polygon?

Now look at a convex pentagon. We can decompose it into triangles, using the "diagonals" from a single vertex, as shown:

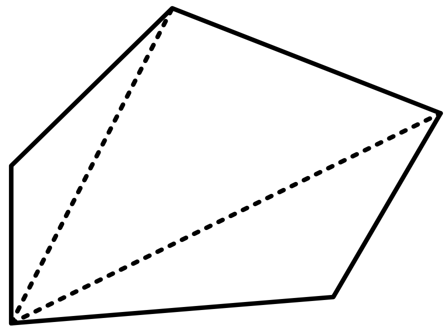


Figure 4.9 Diagonals

We see that in a 5-gon we get three such triangles. Again, the angles of the triangles make up the angles of the polygon when it is decomposed in this way. What is the sum of the measures of all the angles in the triangles? What is the sum of the measures of all the angles in the polygon?

Repeat this procedure with a hexagon to find the sum of the measures of its angle. Continue the process until you can state a formula which will give the sum of the measures of the angles of an n-gon in terms of n. If you have succeeded, you will have found the next theorem.

Theorem!
4.1. The sum of the angles of a polygon with n sides is

$$(n - 2).180°$$

Proof!
Let. $P_1, P_2, \ldots, P_n$ be the vertices of the polygon as shown in the figure. The segments

$$\overline{P_1P_3}, \quad \overline{P_1P_4}, \quad \cdots, \quad \overline{P_1P_{n-1}}$$

decompose the polygon into (n – 2) triangles. Since the sum of the angles of a triangle has 180°, it follows that the sum of the angles of the polygon has (n – 2).180°. □

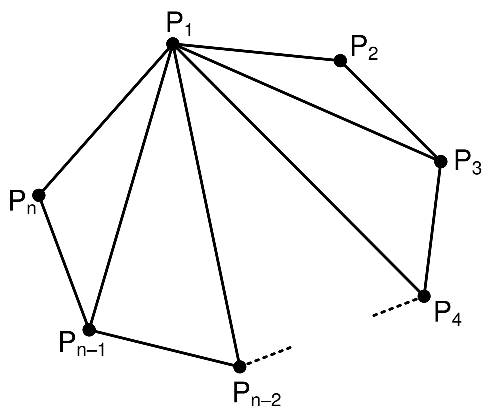


Figure 4.10 Condition of th. 4.1

That's it some basic properties of polygon. In next sub chapter, we will learn more about the properties of polygon.

C. BASIC THEOREMS OF POLYGONS

Theorem!
4.2. In the n-gon, we can build (n – 3) diagonals from a vertex.

Proof!
Look at figure below.

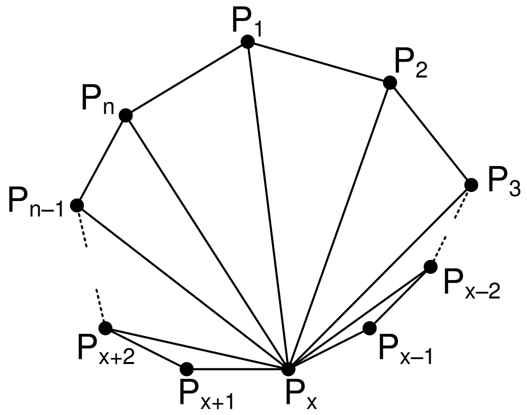


Figure 4.11 Condition of th. 4.2

Let. P_1, P_2 , … ,P_n be the vertices of the polygon as shown in the figure.
Take any vertex as the certain vertex, let it be P_x.

The segments

$$\overline{P_xP_1}\ ,\ \overline{P_xP_2}\ ,\ \cdots\ ,\ \overline{P_xP_{x-2}}\ ,\ \overline{P_xP_{x+2}}\ ,\ \cdots\ ,\ \overline{P_xP_{n-1}}\ ,\ \overline{P_xP_n}$$

build diagonals. So that there are 2 vertices those could not be connected from a chosen vertex, that is P_{x-1} , and P_{x+1} which are not build diagonals.
Obvious P_x couldn't make diagonal to itself.
Because of P_x is any vertex, we conclude that the condition is suitable for all vertices.
So we can only build (n – 3) diagonal from a vertex in a n-gon. □

Theorem!
4.3. The sum of diagonal in an n-gon is

$$\frac{1}{2}n(n-3)$$

Proof!
From theorem 4.2 we get there are (n – 3) diagonal for every vertex in an n-gon.
Obvious from P_1 there are (n – 3) diagonals,
from P_2 there are (n – 3) diagonals,

from P_3 there are (n – 3) diagonals,

…

from P_{n-1} there are (n – 3) diagonals, and

from P_n there are (n – 3) diagonals.

Take any P_x vertex in n-gon given.
Obvious there are (n – 3) diagonal out from the vertex and also there are (n – 3) diagonal in from another vertices.
It means every vertex in an n-gon is being counted twice.

So that the sum of diagonal in a n-gon should be

$$\frac{2n.(n-3)}{2} = \frac{1}{2}n.(n-3)$$

These prove the theorem. □

Theorem!
4.4. The sum of exterior angle of an n-gon is 360°.

Proof!
We leave the proof as exercise.

D. REGULAR POLYGON

A polygon is called **regular** if all its sides have the same length and all its angles have the same measure. For example, a square is also a regular 4-gon. An equilateral triangle is a regular 3-gon.

The **perimeter** of a polygon is defined to be **the sum of the lengths of its sides**.

Example!
If each angle of a regular polygon has 135°, how many sides does the polygon have?

Answer.
Let n be the number of sides. This is also the number of vertices.
Since the angle at each vertex has 135°, the sum of these angles has 135n°. By Theorem 4-1[1], we must have

$$135n = (n-2)180.$$

By algebra, this is equivalent with

$$135n = 180n - 360,$$

and we can solve for n. We get

[1] *Th. 4.1 The sum of the angles of a polygon with n sides is (n – 2).180°*

$$360 = 180n - 135n = 45n,$$

Then we get n = 8.

The answer is that the regular polygon has 8 sides.

E. QUADRILATERAL

Let A, B, C, D be four points which determine the four-sided figure consisting of the four sides AB, BC, CD, and AD. Any four-sided figure in the plane is called a quadrilateral. If the opposite sides of the quadrilateral are parallel, that is, if AB is parallel to CD and AD is parallel to BC, and then the figure is called a **parallelogram**:

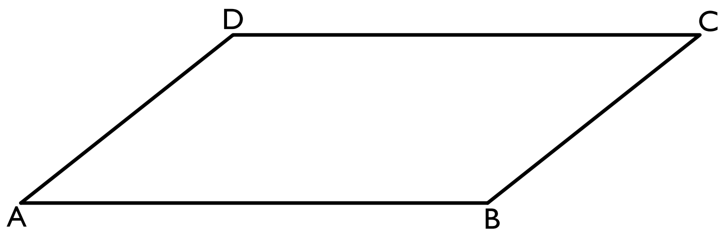


Figure 4.12 A parallelogram

A rectangle is a quadrilateral with four right angles, as shown in figure below.

Figure 4.13 A rectangle

In the definition of a rectangle we only assumed something about the angles. The following basic properties will be used constantly.

Basic Property 1! The opposite sides of a rectangle are parallel, and therefore a rectangle is also a parallelogram. ***Basic Property 2!*** The opposite sides of a rectangle have the same length.

It is also true that the opposite sides of a parallelogram have the same length, but to prove this we need additional axioms. A **square** is a rectangle all of whose sides have the same length. More detailed information to the topics is shown below.

1. Parallelogram

Definition!
4.1. Parallelogram is a quadrilateral which its corresponding sides are parallel.

Theorem!
4.5. In a parallelogram, the size of opposite angles is same.

Proof!
Look at the figure below.

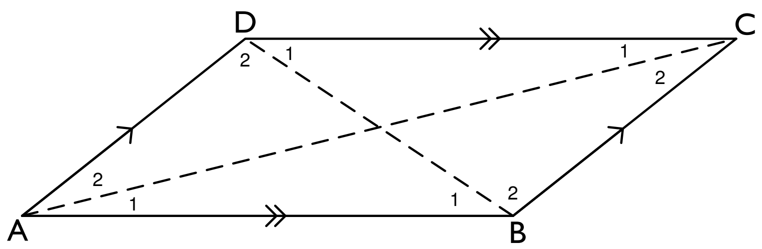


Figure 4.14 A parallelogram with its some parts

It is given a parallelogram ABCD.
Prove that $m\angle A = m\angle C$ and $m\angle B = m\angle D$.
Draw diagonal AC.
Obvious $m\angle A_1 = m\angle C_1$ (**alternate interior angles**) and
$m\angle A_2 = m\angle C_2$ (**alternate interior angles**).
We get $m\angle A_1 + m\angle A_2 = m\angle C_1 + m\angle C_2$
$\Leftrightarrow m\angle A = m\angle C$.
By the similar step, we get $m\angle B = m\angle D$.
So the size of opposite angles in a parallelogram is same. □

Theorem!
4.6. In a parallelogram, the length of opposite sides is same.

Proof!
Look at the figure 3.12.

It is given a parallelogram ABCD.
Prove that AB = CD and BC = DA.

Draw diagonal AC.
Look at ΔABC and ΔACD
Obvious $m\angle A_1 = m\angle C_1$ (*because of* ***alternate interior angles***),
$m\angle A_2 = m\angle C_2$ (*because of* ***alternate interior angles***),
AC = AC (because of coincide).
We get $\Delta ABC \cong \Delta ACD$ (because of ASA)
One of the consequence is AB = CD and BC = AD. □

Theorem!

4.7. In a parallelogram, both of its diagonals are intersecting in the midpoint of its diagonals.

Proof!

Look at figure below.

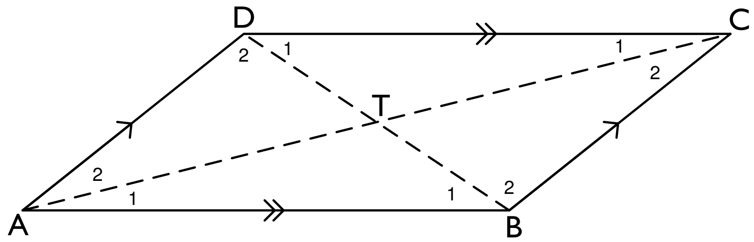


Figure 4.15 A parallelogram with its some parts

It is given a parallelogram ABCD.

Prove that AT = TC and BT = TD.

Look at ΔATD and ΔBTC.

Obvious $m\angle D_2 = m\angle B_2$ (*because of* ***alternate interior angles***),

$m\angle A_2 = m\angle C_2$ (*because of* ***alternate interior angles***),

AD = BC (because of theorem 3.6).

We get Δ ATD ≅ ΔBTC (because of ASA)

One of the consequence is AT = TC and BT = TD.

So that T is midpoint of AC and T is also midpoint of BD. □

Theorem!

4.8. In a quadrilateral, if the size of opposite angles is same, then the quadrilateral given is a parallelogram.

Proof!

Look at the figure below.

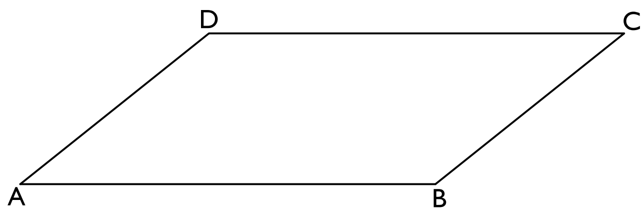


Figure 4.16

It is given a quadrilateral ABCD and

$m\angle A = m\angle C$ also $m\angle B = m\angle D$.

Prove that ABCD is a parallelogram.

Obvious the sum of all angles in 4-gon is (4 – 2).180° = 360°. (based on th. 4.1)

Obvious $m\angle A + m\angle B + m\angle C + m\angle D = 360°$

$\Leftrightarrow$ $m\angle A + m\angle B + m\angle A + m\angle B = 360°$

$\Leftrightarrow$ $2.\ m\angle A + 2.m\angle B = 360°$

$\Leftrightarrow$ $m\angle A + m\angle B = 180°$.

We get:

1. Because of $m\angle A + m\angle B = 180°$ then AD || BC.
2. Because of $m\angle A + m\angle B = 180°$ and $m\angle B = m\angle D$ then $m\angle A + m\angle D = 180°$.
3. Because of $m\angle A + m\angle D = 180°$ then AB || CD.

Because AD || BC and AB || CD, we conclude that quadrilateral ABCD is a parallelogram. □

Theorems!

4.9. In a quadrilateral, if the length of opposite sides is same, then the quadrilateral given is a parallelogram.

4.10. In a quadrilateral, if both of diagonals are intersecting one another at the midpoint, then the quadrilateral given is a parallelogram.

4.11. In a quadrilateral, if there are two pair's parallel sides, then quadrilateral given is a parallelogram.

Proof!

We leave the proof for 4.9, 4.10, and 4.11 as exercises.

2. Rectangle

Definition!

4.2. A rectangle is a parallelogram which one of its angle is a right angle.

To help you to be more understood, see figure 4.17 below.

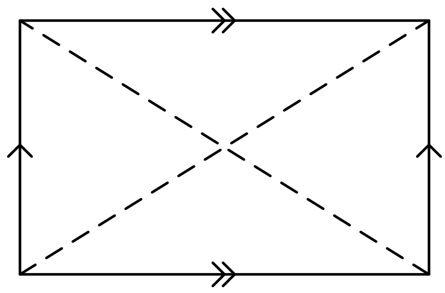


Figure 4.17 A rectangle with diagonals

The consequences are:

a. A rectangle has four right angles.
b. All properties of parallelogram also obtain to rectangle.

Theorem!
4.12. In a rectangle, the length of its diagonals is same.

Proof!

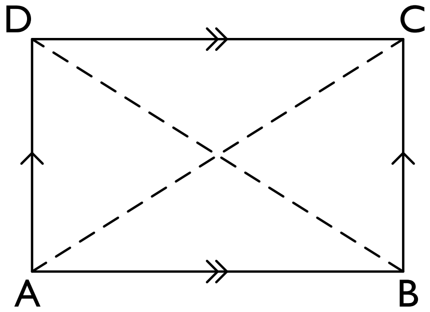


Figure 4.18 Rectangle with label

Look at figure 4.18.
It is given rectangle ABCD.
Prove that AC = BD.

Look at ΔABD and ΔABC.
Obvious
1. m∠B = m∠A (90°),
2. BC = AD (property of parallelogram hold on rectangle), and
3. AB = AB (coincide).

From 1, 2, 3, we conclude that ΔABD ≅ ΔABC.

One of the consequence is |AD| = |BC|. □

Theorem!
4.13. In a parallelogram, if the length of the diagonals is same then the parallelogram given is a rectangle.

Proof!
We leave the proof as exercise to the reader.

3. **Rhombus**

Definition!
4.3. A rhombus is a parallelogram which has property the length of two adjacent sides is same.

To help you to be more understood, see figure 4.19 below.

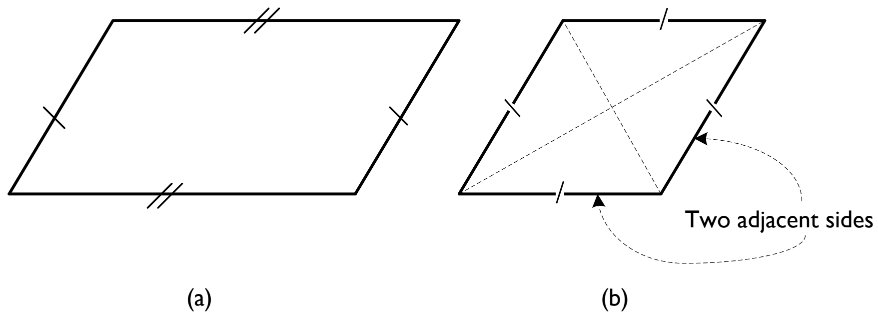


Figure 4.19 Rhombus compare with a parallelogram

Figure 4.19 (a) show us the parallelogram, and 4.19 (b) show us parallelogram with addition properties. Figure 4.19 (b) show us the rhombus.

The consequences of parallelogram properties held are:
a. In the rhombus, all sides have similar length.
b. All properties of parallelogram also obtain to rhombus.

Theorem!
4.14. The diagonals of rhombus, bisect the angle be similar size and the diagonals are perpendicular each other.

Proof!

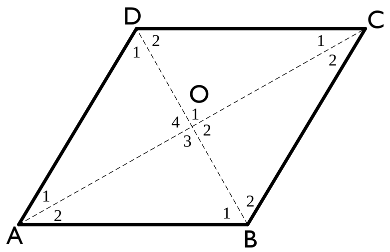


Figure 4.20 Rhombus with its special lines

It is given rhombus ABCD.
Prove that :
i. $m\angle A_1 = m\angle A_2$; $m\angle B_1 = m\angle B_2$; $m\angle C_1 = m\angle C_2$; $m\angle D_1 = m\angle D_2$.
ii. $AC \perp BD$.

We will proof (i).

Obvious ΔABC and ΔACD are isosceles triangles.
Obvious
a. $m\angle A_1 = m\angle C_2$ (because of alternate interior angle),
b. $m\angle A_2 = m\angle C_1$ (because of alternate interior angle),
c. $m\angle A_2 = m\angle C_2$ (because of ΔABC is an isosceles triangles), and
d. $m\angle A_1 = m\angle C_1$ (because of ΔACD is an isosceles triangles).

Because of $m\angle A_1 = m\angle C_2$ and $m\angle A_2 = m\angle C_2$ we conclude that $m\angle A_2 = m\angle A_1$.

Because of $m\angle C_1 = m\angle A_2$ and $m\angle C_2 = m\angle A_2$ we conclude that $m\angle C_2 = m\angle C_1$. We get AC is bisect $\angle A$ and $\angle C$ into two similar size angles.

By similar steps, we can proof that BD is bisect $\angle B$ and $\angle D$ into two similar sizes.

Now we will proof (ii).

Look at ΔODC and ΔOBC.
Obvious

a. BC = CD (property of rhombus)
b. $m\angle C_2 = m\angle C_1$ (from (i))
c. OC = OC (coincide)

From a, b, and c, we conclude that $\Delta ODC \cong \Delta OBC$ (because SAS)

The consequence is $m\angle O_1 = m\angle O_2$.
Obvious ΔBCD is an isosceles triangle.
Obvious $m\angle O_1 = m\angle C_1 + m\angle D_2$, from ΔOCD.
Obvious $m\angle B_2 + m\angle C + m\angle D_2 = 180°$.
Because of $m\angle B_2 = m\angle D_2$, then we get
$m\angle B_2 + m\angle C + m\angle D_2 = 180°$
$\Leftrightarrow 2 \,.\, m\angle D_2 + m\angle C = 180°$
$\Leftrightarrow m\angle D_2 + \frac{1}{2}.m\angle C = 90°$
$\Leftrightarrow 180° - m\angle O_1 = 90°$.
So that $m\angle O_1 = m\angle O_2 = 90°$.
So $AC \perp BD$. □

Theorems!

4.15. In a parallelogram, if a diagonal bisect an angle into two similar sizes then the parallelogram is a rhombus.
4.16. In a parallelogram, if the diagonals are perpendicular each other then the parallelogram is a rhombus.

Proof!
We leave the proof as exercise to the reader.

4. **Square**

Definition!
4.4. A square is a quadrilateral which it's all sides has similar length and one of its angles is right angle.

To help you to be more understood, see figure 4.21 below.

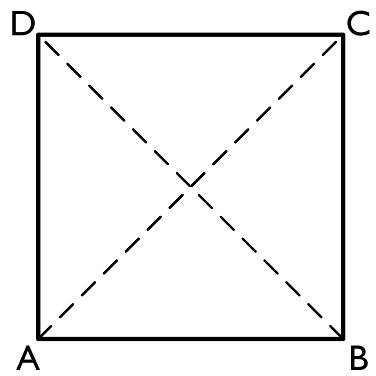


Figure 4.21 A Rectangle

The direct consequences are:
a. All angles in square are right angle.
b. All properties of rhombus and rectangle also obtain to square.

Square is also called regular quadrilateral.

5. **Trapezoid**

Definition!
4.5. A trapezoid is a quadrilateral which has exactly one parallel side.

To help you to be more understood, see figure 4.22 below.

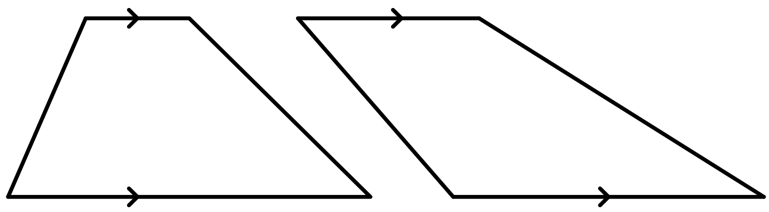


Figure 4.22 Trapezoid in another forms

Trapezoid which has similar lateral side is called isosceles trapezoid.

Theorems!
4.17. In the isosceles trapezoid, the base angles have similar size.

Proof!

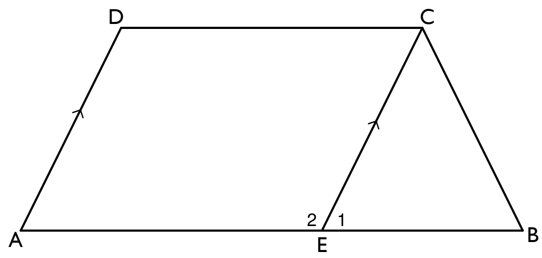


Figure 4.23 Condition of th. 4.17

It is given isosceles trapezoid ABCD (AD = BC)
Prove that m∠A = m∠B.

Draw line CE such that AD ∥ CE.
Obvious AECD is a parallelogram and AD = CE.
The consequence is ΔBEC is an isosceles triangle (because of AD = CE and AD = BC).
Then we get m∠E_1 = m∠B.
Because of m∠A = m∠E_1 (corresponding angle), we conclude that m∠A = m∠B. □

Theorems!

4.18. In a trapezoid, if the base angles have similar size then the trapezoid is an isosceles trapezoid.

4.19. In a trapezoid, if the diagonal has same length then trapezoid is an isosceles trapezoid.

Proof!
We leave the proof as exercise for the reader.

6. **Kite**

Definition!
4.6. A kite is a quadrilateral which has a pair of similar length adjacent side and perpendicular diagonals.

To help you to be more understood, see figure 4.22 below.

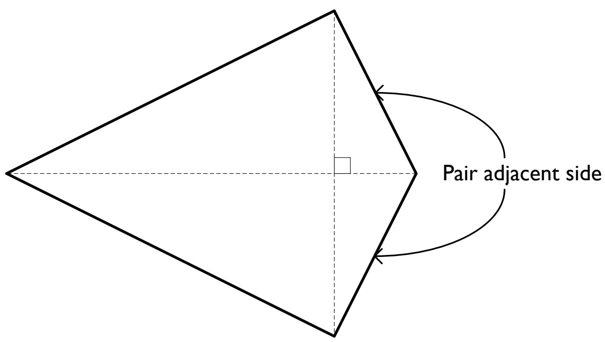


Figure 4.24 A kite with pair of adjacent sides

We have given a structure like **Contrast Theorem**. Contrast is one way to check your understanding. If the problem at contrast could be solved, we will sure that the knowledge are transferred well.

Contrast theorem is just like giving sample that is not true about the topics. But the challenge is which sample is it. It is hard to find the contrast. Not all topics can be found the contrast easily. Sometimes we have to discuss deeply just only to find one good contrast problem.

Sample of contrast is given below.

Look at the figure below!

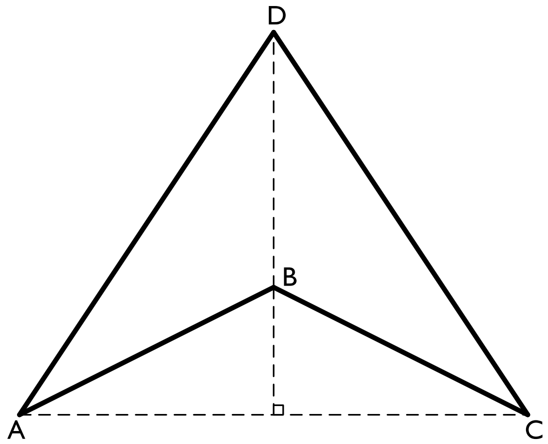


Figure 4.25 Is the planar above kite?

Is figure 4.25 a kite?
Explain your answer!

Hints!
Please read sub chapter about convexity.

That's all explanation about quadrilateral. Some problems about the topics were given below.

F. EXERCISE CHAPTER 4 #1

1. In the 4-gon ABCD, S is an intersection point of the diagonals. Let AS = SC and AB || DC. Prove that the 4-gon is a parallelogram!
2. Let ABCD is a square. On AB take any vertex P, on BC take any vertex Q, on CD take any vertex R, and on AD take any vertex S, such that AP = BQ = CR = DS. Prove that PQRS is a square!
3. In the rhombus ABCD, the line m from B is perpendicular to AD bisecting the side AD in similar length. Find the size of all rhombus angles!

4. It is known a 4-gn ABCD. The midpoints of AB, CD, BD, and AC are P, Q, R, S. Prove that PQ and RS intersecting at midpoint!

5. It is known a parallelogram ABCD. P and Q is midpoint of AB and CD. Prove that AQ and CP dividing diagonal BD be three equal parts!

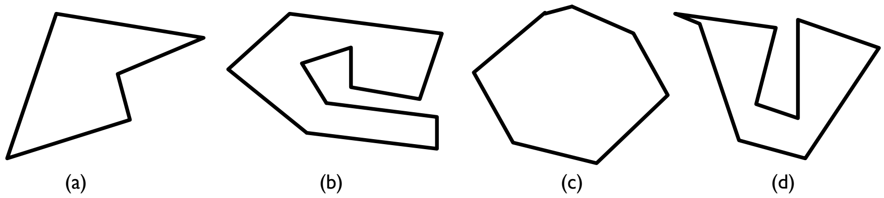
6. Prove that the line which passes through the midpoint of parallel side of isosceles trapezoid is perpendicular to its parallel side!

7. In the trapezoid ABCD (AB // CD), P and Q are midpoints of AD and BC. The line PQ intersect AC at R and BD at S. Prove that PR = QS!

8. Prove this theorem. The midpoints of isosceles trapezoid side are a vertex of a rhombus.

G. EXERCISE CHAPTER 4 #2

1. Determine whether or not each of the following figures is a polygon. If it is, state whether or not it is convex:



2. What is the sum of the measures of the angles of:
   (a) an octagon (b) a pentagon (c) a 12-gon

3. Find the measure of each angle of a regular polygon with:
   (a) 6 sides (b) 11 sides (c) 14 sides (d) *n* sides

4. Is it possible to have a regular polygon each angle of which has 153°? Give reasons for your answer.

5. If each angle of a regular polygon has 165°, how many sides does the polygon have?

6. How many sides does a polygon have if the sum of the measures of the angles is:
   (a) 2700° (b) 1080° (c) d^o

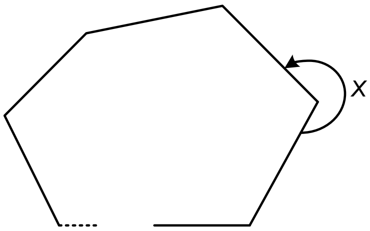
7. If each angle of a regular polygon has 140°, how many sides does the polygon have?

8. Give an example of a polygon whose sides all have the same length, but which is not a regular polygon. Give an example of a polygon whose angles all have the same measure, but which is not a regular polygon.

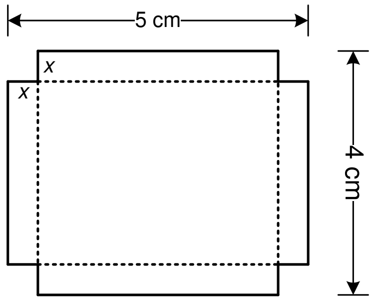
9. An isosceles triangle has a side of length 10, and a side of length 4. What is its perimeter? (Yes, there is enough information given.)

10. The sides of a triangle have lengths 2n - 1, n + 5, 3n - 8 units.
    (a) Find a value of n for which the triangle is isosceles.
    (b) How many values of n are there which make the triangle isosceles?
    (c) Is there a value of n which makes the triangle equilateral?

11. An equilateral triangle has perimeter 36. Find its area.
12. A square has perimeter 36. Find its area.
13. In the figure, angle X is measured on the outside of an arbitrary n-gon. Prove that the sum of all n such outside angles equals (n + 2).180°. *[Hint: What is the sum of all n interior angles?]*



14. Squares of side length x are cut out of the corners of a 4 cm x 5 cm piece of sheet metal, as illustrated:



Show that the perimeter of the piece of metal stays constant, regardless of the value of x, as long as x < 2.

15. What is the area of the piece of metal in Exercise 14?

16. Prove that the area of a regular hexagon with side of length s is:

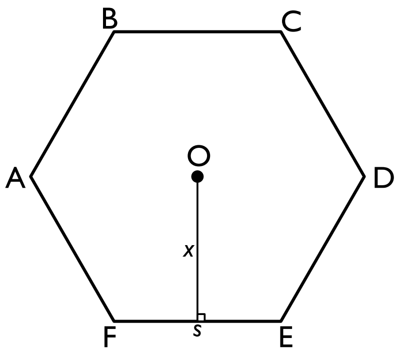
$$\frac{3s^2\sqrt{3}}{2}$$

[*Hint: Divide the hexagon into triangles by drawing "radii". What kind of triangles are they?*]

17. What is the area of a regular hexagon whose perimeter is 30 cm?

18. The length of each side of a regular hexagon is 2. Find the area of the hexagon.

19. In the figure, ABCDEF is a regular hexagon. The distance from the center 0 to any side is x; the length of each side is s. Prove that the area of the hexagon is 3xs.



Chapter 5

THE AREA AND PHYTAGORAS THEOREM

We have talk about some concept of area in chapters before. When we talk about the median we have talk that the median divide a triangle became two parts which each part has same area. In that chapter we just justify my statement is true without proofing. Now, some of our discussion is repeated and maybe we will give you deeper understanding about the area. We begin with the area of a triangle including the area of square, as base, and rectangle.

A. INTRODUCTION

Whenever we choose a unit of distance to measure the length of a line segment (as we did in the first chapter), we are also selecting a unit of area, with which we can measure the amount of space enclosed by a region in the plane. For example, if we select the centimeter as the unit of distance, the corresponding unit of area is the square centimeter. One **square centimeter** is defined as the area of the region enclosed by a square with sides of length one centimeter:

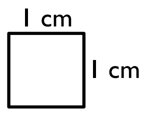


Figure 5.1. Area of 1 cm square

Now, we can find the area of various regions by determining how many unit area squares fit into the region. For example, suppose we have a square with sides of length 4 cm; we see that its area is 16 square cm:

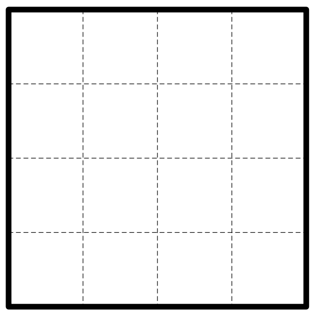


Figure 5.2. Area of 4 cm side square is 16 area units.

Suppose that there are a square which has $2\frac{1}{2}$ cm side length. Please count the number of area unit on the figure below include the fraction.

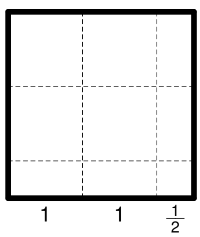


Figure 5.3. Area of 2,5 cm side square.

From figure above we get:
there are 4 full squares, this means the area should be 4 cm square,
there are 4 half squares, this means the area should be 2 cm square, and
there are 1 a quarter square, this means the area should be $\frac{1}{4}$ cm square.
Total area is $6\frac{1}{4}$ cm square.

Project 5.1

1. *Draw a picture illustrating a square whose sides have length 10,25 cm. Determine its area by counting unit squares and fractions of unit squares!*
2. *Draw a rectangle whose sides have lengths 3 cm and 5 cm. Determine its area by counting unit squares!*
3. *Draw a rectangle whose sides have lengths 9,5 cm and 5 cm. Count squares to determine its area!*
4. *In your picture for Problem 2, draw a diagonal of the rectangle (a line segment joining opposite corners), which creates two right triangles.*
   (a) How do the areas of these two triangles compare? Why?
   (b) Compute the areas of these triangles without counting.
5. *Look at figure below.*

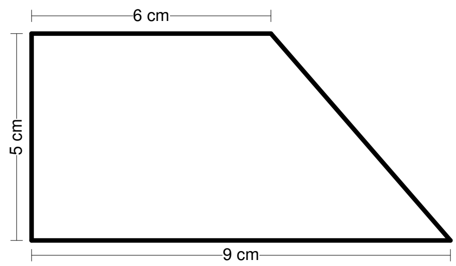


Figure 5.4. A polygon.

Using whatever method please find the area of figure above!

B. THE AREA

From project 5.1 we define the area of a rectangle and square as shown below.

5.1. Definition!
The area of square and rectangle!
Area of a square whose sides have length s is $\mathbf{s \cdot s = s^2}$ square units. Area of a rectangle whose sides have lengths **a, b** units is **(a.b)** square units.

We denote area of ABCD by $\mathbf{A_{ABCD}}$

We know that square has same length for all sides and rectangle has a pair of same length sides. In rectangle we can say, the longer side by **length** and the shorter side by **width**. It is a little bit different with a triangle.

We build a triangle by make a diagonal from a rectangle. See figure below.

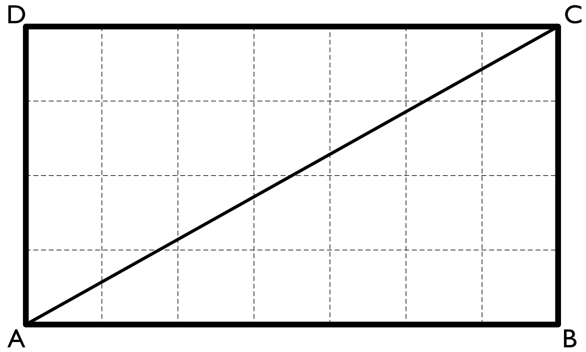


Figure 5.5. Construction of a triangle.

Look at ΔABC and ΔADC.
Obvious $m\angle ACD = m\angle BAC$, because of alternate interior angle,
$|CD| = |AB|$, because of properties of rectangle, and
$m\angle ADC = m\angle ABC$, because of right angle (90°).
We conclude that $\Delta ABC \cong \Delta ADC$.
So that the area of rectangle ABCD could be say twice of the area ΔABC, or we can say that the area ΔABC is a half of the area rectangle ABCD.
We knew that the area of rectangle ABCD is 28 area unit, from AB × BC. So we can write, the area of ΔABC is

$$\frac{1}{2} . AB \times BC \qquad \ldots\ldots\ldots 1*$$

If AB or length of rectangle can be expressed by **base** of ΔABC, and BC or width of rectangle ABCD can be expressed by the **altitude** of ΔABC, then the formula on 1∗ can be expressed by

$$\mathbf{A_{\Delta ABC} = \frac{1}{2} . base \times altitude} \qquad \ldots\ldots\ldots 2*$$

From 2∗ we may write the definition below.

5.2. Definition
The area of triangle!
Area of a triangle whose base (b) and altitude (a) are known is $\frac{1}{2}$ **. b** × **a** square units.

We denote area of triangle by $\mathbf{A_{\Delta ABC} = \frac{1}{2} . b \times a}$

We will proof that the formula in definition 5.2 is hold for all kinds of triangle. See figure below.

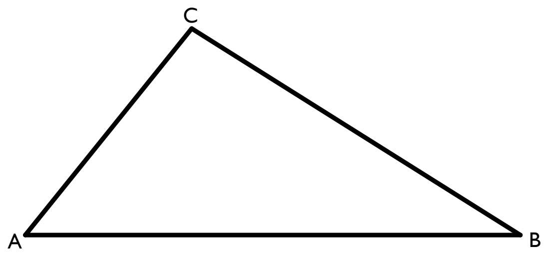


Figure 5.6. A scalene triangle.

This proof for scalene triangle is also held for isosceles triangle and equilateral triangle. Generally, this proof is held for acute triangle.

Draw the altitude from C to opposite side of C, give label for intersection point of altitude and base with F! Now, find the midpoint of AC and BC! Label it with D and E! We assume the line DE divide the altitude CF in to two similar parts. *(Why?)*

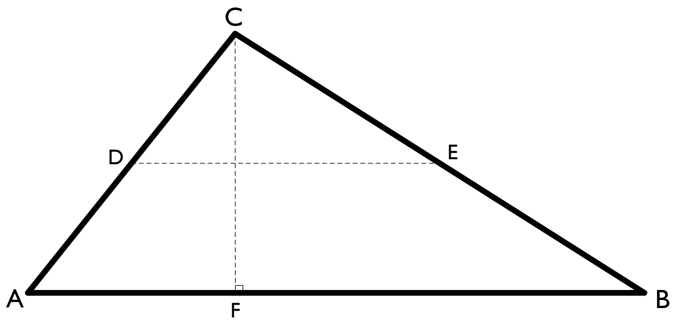


Figure 5.7. Proof of the area of triangle. STEP #1

Extend the line DE! Draw a perpendicular line from vertex A and B such that cross the extension line of DE.

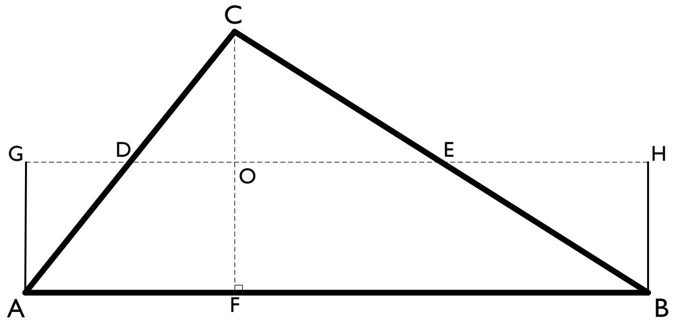


Figure 5.8. Proof of the area of traingle. STEP #2

By the properties of congruency, you may proof that $\Delta AGD \cong \Delta DOC$, and also $\Delta BEH \cong \Delta COE$. *(Proof that!)* So we assume that the area of ΔABC equals to the area of rectangle ABHG. We know that the area of rectangle ABHG,

$$A_{ABHG} = AB \times AG = AB \times FO = AB \times \frac{1}{2}.CF = \frac{1}{2} \times AB \times CF$$

We get that $A_{\Delta ABC} = A_{ABHG} = \frac{1}{2} \times AB \times CF$. Obvious AB is base of triangle and CF is the altitude of ΔABC. So we conclude that the area of ΔABC equal $\frac{1}{2} \times$ base $\times$ the altitude. What if the triangle given is an obtuse triangle as shown below?

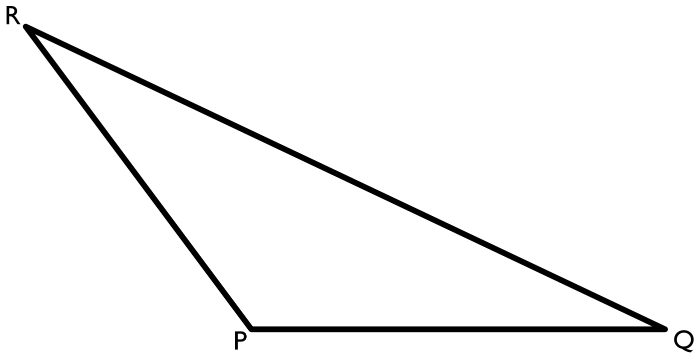


Figure 5.9. An Obtuse Triangle

To proof formula 2∗ also applied for obtuse triangle, we will show first the formula of parallelogram below. Before we explain the formula of a parallelogram, you may find the area of figure below by whatever the way!

Figure 5.10. A Parallelogram

Draw perpendicular lines from N down to it's base, and also from M, until intersects the extension of KL.

Figure 5.11. Proofing the area of parallelogram

From figure 5.11 above we assume that ΔKSN ≅ ΔLTM. *(Why? Proof that!).* So we can eliminate and move ΔKSN and coinciding to ΔLTM. We get the area of parallelogram KLMN is equal to the area of rectangle STMN, that is ST × NS. Because of |ST| = |KL|, we may write the area of STMN by KL × NS.

5.3. Definition

The area of parallelogram!

Area of a parallelogram whose base (b) and altitude (a) are known is **b × a** square units.

We denote area of parallelogram KLMN by

$\mathbf{A_{KLMN} = b \times a}$

Now, we proof the formula in definition 5.2 also hold for obtuse triangle on figure 5.9. Let draw a congruent triangle PQR, give label P'QR, and rotate it 180°. After that please coincide the side QR each other.

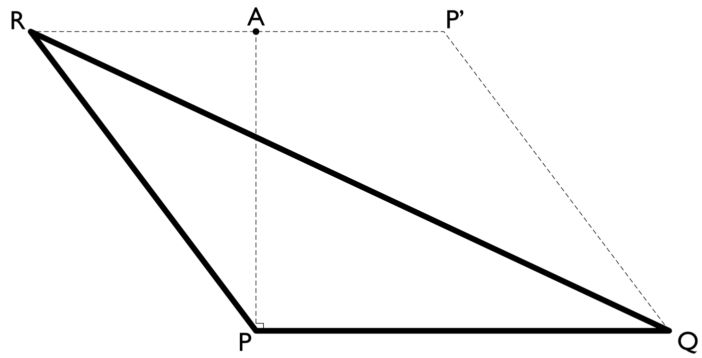


Figure 5.12. Proofing the area of obtuse triangle. #1

Figure 5.12 show us two obtuse triangles builds parallelogram PQP'R. We knew that $A_{PQP'R} = PQ \times AP$. We get $A_{\Delta PQR} = \frac{1}{2} \times PQ \times AP$. That is $\frac{1}{2}$ of base times it's altitude. Here we give you a figure which shows another proof of obtuse triangle.

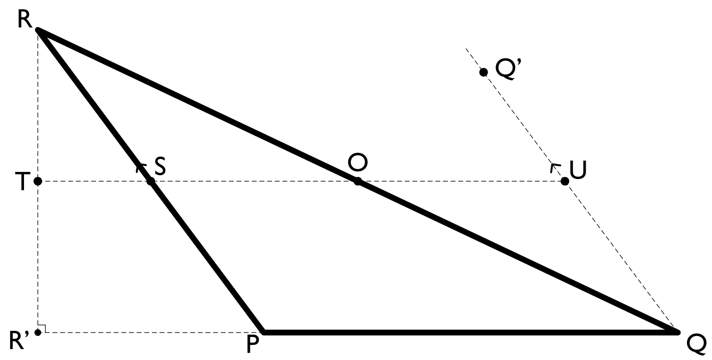


Figure 5.13. Proofing the area of obtuse triangle. #2

First, draw a line QQ' parallel with PR. From the midpoint of altitude RR', draw line parallel with PQ such that intersect PR at S, QR at O and QQ' at U.

We get ΔRSO ≅ ΔQOU. *(Why? Proof that!)*

So the area of triangle PQR is equal with the area of parallelogram PQUS, that is PQ × R'T. Now, we can write

$$A_{\Delta PQR} = R'T \times PQ = \frac{1}{2} \times RR' \times PQ = \frac{1}{2} \times \textit{altitude} \times \textit{base}$$

We have proof that the formula in 5.2 can be applied to any triangle given.

Back to project 5.1! Now we will explain how to find the area of a trapezoid as shown on figure 5.4 above. Let there are trapezoid as shown below.

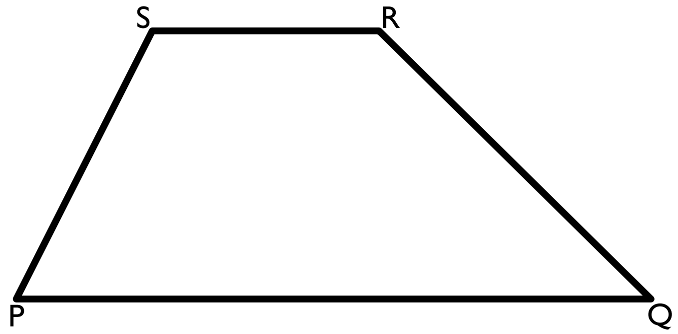


Figure 5.14. Inventing the area of trapezoid.

Draw altitude from vertex S, P, and Q, and then find it's midpoint, label it with T! After that draw line through T parallel to PQ until cross PP' and QQ'! (PP' and QQ' are altitude through each P and Q). See figure below!

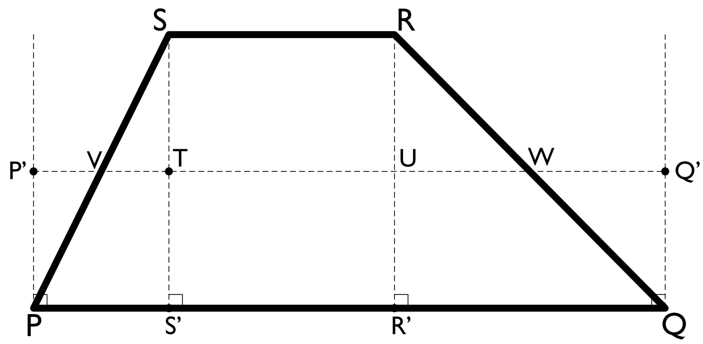


Figure 5.15. Inventing the area of trapezoid.

Let we denote the altitude by a, so that from figure above we get area of trapezoid,

$$
\begin{aligned}
A_{PQRS} &= A_{PP'TS'} + 2 \,.\, A_{TURS} + A_{QQ'UR'} \\
&= \tfrac{1}{2} \,.\, a \,.\, PS' + 2 \,.\, \tfrac{1}{2} \,.\, a \,.\, S'R' + \tfrac{1}{2} \,.\, a \,.\, R'Q \\
&= \tfrac{1}{2} \,.\, a \,.\, (PS' + 2.\, S'R' + R'Q) \\
&= \tfrac{1}{2} \,.\, a \,.\, (PQ + S'R') \\
&= \tfrac{1}{2} \,.\, a \,.\, (PQ + SR)
\end{aligned}
$$

Because of PQ and RS are parallel sides of trapezoid, we may write definition below.

5.4. Definition!

The area of trapezoid!

Area of a parallelogram whose altitude (a) and two parallel side p and q are known is $\frac{1}{2} \times \mathbf{a} \times \mathbf{(p + q)}$ square units.

We denote area of trapezoid PQRS by

$$\mathbf{A_{PQRS} = \frac{1}{2} \,.\, a \,.\, (PQ + SR)}$$

We let the reader to proof two trapezoid below are suitable with the formula in definition 5.4.

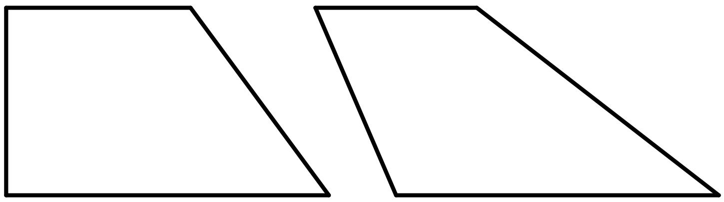


Figure 5.16. Proof that trapezoids above are suitable to formula in def. 5.4.

Another quadrilateral is a rhombus and kite. We may justify that to find the area of a rhombus is similar to find the area of parallelogram. We sure it is easier. But we give some another proof of it. See figure below.

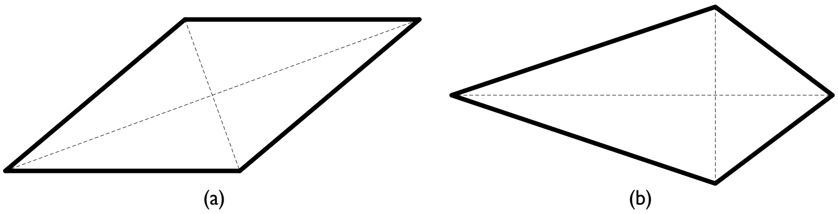


Figure 5.17. Rhombus and a kite

We will proof only for kite. To find the formula of a rhombus, please do yourself as the exercise. Let see figure 5.17 (b), please give it label KLMN. See figure below.

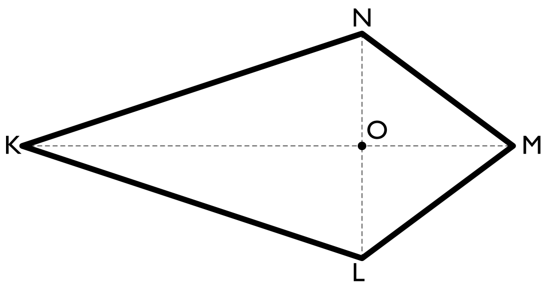


Figure 5.18. Rhombus

Draw lines through K parallel with LN. Do it from vertex M. And after that draw line parallel to KM through L!

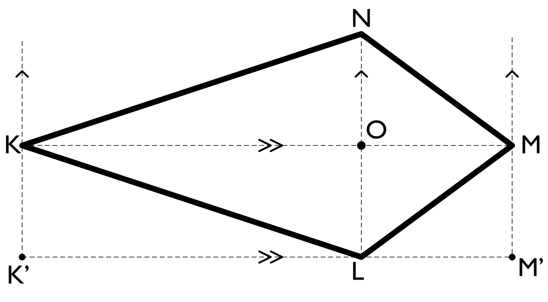


Figure 5.19. Inventing the area of Kite.

We justify that ΔKNO ≅ ΔKK'L and ΔMNO ≅ ΔMM'L *(Why?)*. So that, the area of kite KLMN equals with the area of rectangle KMM'K'.
We can write the formula of area of a kite,

$$
\begin{aligned}
A_{KLMN} &= A_{KMM'K'} \\
&= |KM| \times |MM'| \\
&= |KM| \times |OL| \\
&= |KM| \times \tfrac{1}{2} \,.\, |LN| \\
&= \tfrac{1}{2} \,.\, |KM| \,.\, |LN|
\end{aligned}
$$

Please realize that KM and LN are diagonals. So we can write definition below.

> **5.5. Definition!**
> ***The area of Rhombus and Kite!***
> Area of a rhombus and trapezoid whose it's diagonal (d_1) and diagonal (d_2) are known is $\frac{1}{2} \times \mathbf{d_1} \times \mathbf{d_2}$ square units.
>
> **We denote area of Rhombus and Kite PQRS by**
> $$\mathbf{A_{PQRS} = \tfrac{1}{2} \,.\, \times d_1 \times d_2}$$

Now, give attention to figure 5.17! What can you conclude about both of quadrilateral? See deeper to its diagonal. After that we can give a property as shown below.

Theorem!

5.1. The area of quadrilateral which its diagonals are perpendicular each other is a half of multiplication it's diagonals length.

See figure below for more detail explanation.

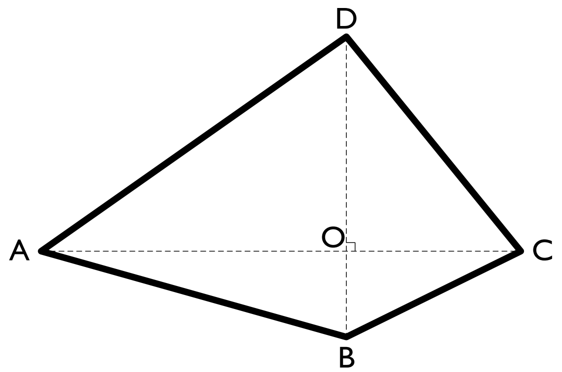


Figure 5.20. Condition of theorem 5.1.

It is given a 4-gon ABCD. The theorem is just tell us that

$$\mathbf{A_{ABCD} = \tfrac{1}{2} \times |AC| \times |BD|}$$

Proof!

We give two kinds of proofing here, analytically and geometrically. We let you choose which is easier to understand.

Proof #1!

Obvious $A_{ABCD} = A_{\Delta ACD} + A_{\Delta ACB}$

$= \left(\frac{1}{2} \times AC \times OD\right) + \left(\frac{1}{2} \times AC \times OB\right)$

$= \frac{1}{2} \times AC \times (OD + OB)$

$= \frac{1}{2} \times AC \times BD$

We conclude that $A_{ABCD} = \frac{1}{2} \times |AC| \times |BD|$. □

Proof #2!

Draw O_1 and O_2 midpoints of OD and OB! Draw lines parallel with AC through O_1 and another parallel line through O_2. See figure below.

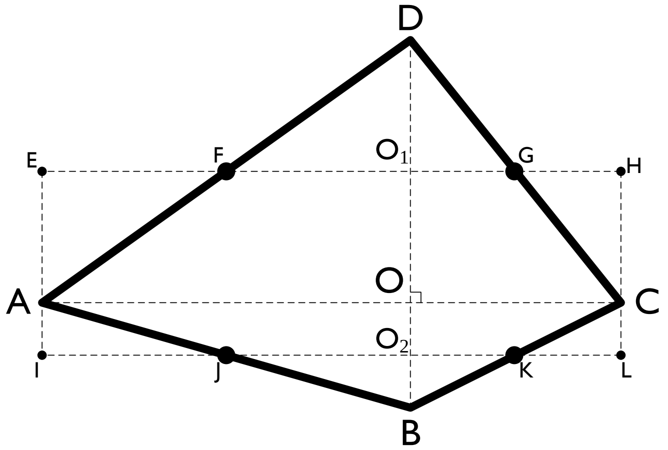


Figure 5.21. Proof of theorem 5.1.

We get $\Delta DFO_1 \cong \Delta AEF$; $\Delta DGO_1 \cong \Delta CGH$; $\Delta AIJ \cong \Delta BO_2J$; and $\Delta CKL \cong \Delta BO_2K$.

We conclude that the area of 4-gon ABCD equal the area of rectangle ILHE that is IL × IE = IL × $\frac{1}{2}$ × BD = $\frac{1}{2}$ × AC × BD.

This proof the theorem. □

C. EXERCISE CHAPTER 5 #1

1. In each of the following, the base *b* and height *h* of a triangle is given. Find the area:
   (a) *b* = 12 cm, *h* = 5 cm (b) *b* = 3 m, *h* = 7 m (c) *b* = 2*x*, *h* = *x*
2. Find a side of a square whose area is equal to the area of a rectangle with sides 10 and 40.
3. The height of a triangle is one half the bases. Find its base if the area of the triangle is 36 sq. m.
4. Find the dimensions of a rectangle whose length is five times its width and whose area is 1440 sq. cm.

5. The piece of sheet metal illustrated below folds into a box with a square bottom and no top:

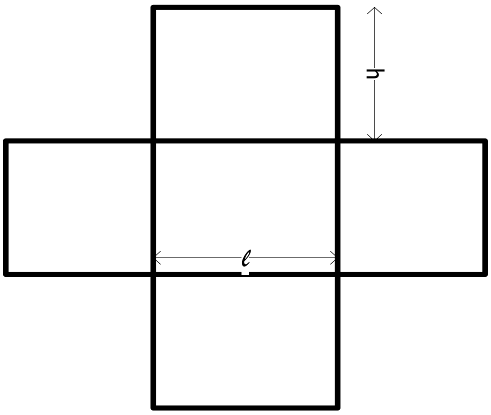


Figure 5.22. Condition of problem 5.

   (a) *How much material is needed (in square cm) if* $\ell = 5$ cm *and* h = 4 cm?

   (b) *Write a formula for the area of the metal in terms of* ℓ *and* h.

6. A man wishes to build a path 1 m wide around a garden which is a rectangle 15 m by 20 m. What is the area of the path?

7. The sides of a rectangle are in the ratio 3: 4 and its area is 300 sq. cm. Find its sides!

8. Find the base of a triangle if its area is 36 sq. cm and its height is 12 cm.

9. A right triangle has legs of equal length and area 40. How long are the legs?

10. In right triangles ΔABC and ΔXYZ below, each leg in ΔXYZ has twice the length of the corresponding leg of ΔABC. What is the ratio of the area of ΔABC to the area of ΔXYZ?

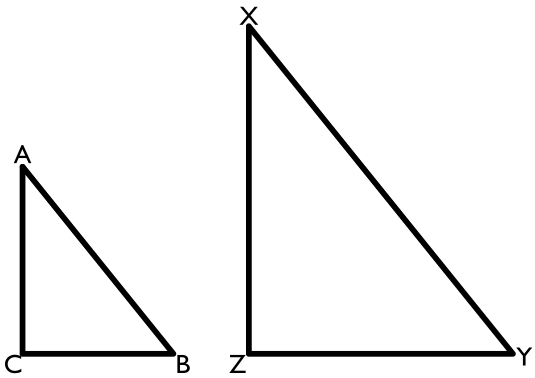


Figure 5.23. Condition of problem 10.

11. In Problem 10, suppose the legs of X Y Z are n times as big as the legs in6.ABC, where n is a positive integer. What is the ratio of the areas of the triangles?

12. The length of a side of a square is 8 units. What is the length of a side of the square whose area is three times as large?

13. A swimming pool measures 6 m by 9 m. You want to cover the floor with tiles which come in squares 0.5 m on a side, and which cost $35 each. How much will this project cost you?

14. Three squares, with sides of length 5, 4, and 3 respectively, are placed together as shown. Find the area of the shaded region.

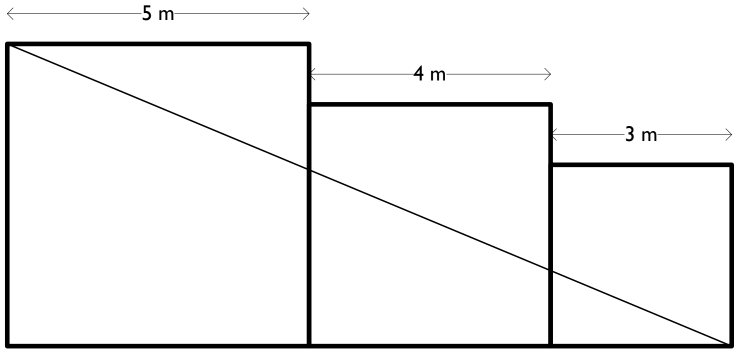


Figure 5.24. Condition of problem 14.

15. If a trapezoid has a base 8 and height 7, what is the length of the other base if the area of the trapezoid is 77.

16. Three city lots, each with 25 m frontage along the main drag, make up a single tract, as illustrated below:

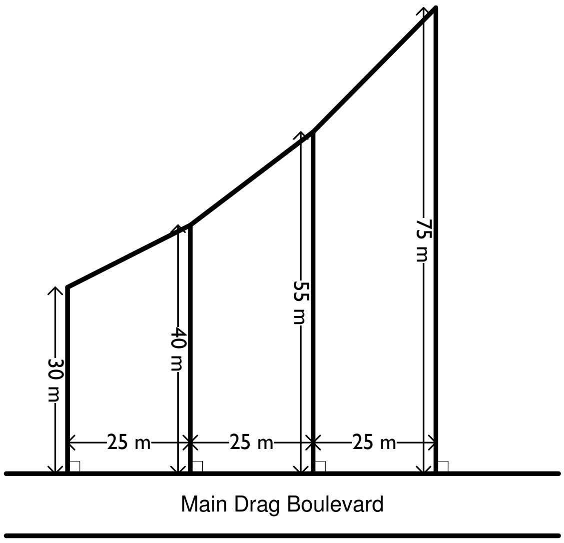


Figure 5.25. Condition of problem 16.

What is the total area of the tract?

17. Let ΔPQM be a triangle, as shown below. Let N be the midpoint of the segment QM. Prove that the triangles ΔPQN and ΔPNM have the same area.

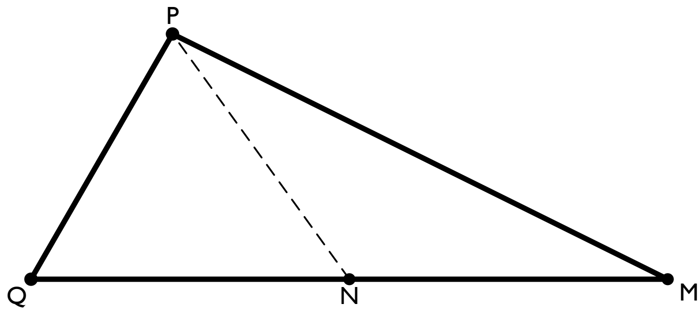


Figure 5.26. Condition of problem 17.

18. Let ΔPQM be a triangle, as shown below. Let N be the point on the segment QM such that the distance from N to M is twice the distance from Q to N, as shown.

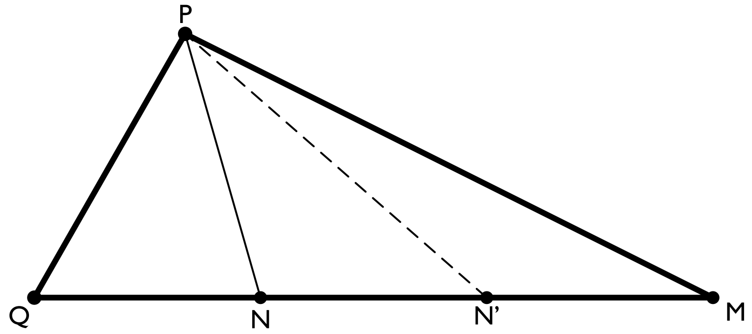


Figure 5.27. Condition of problem 18.

(a) *Prove that the triangle ΔPNM has twice the area as ΔPQN.*
(b) *How does the area of ΔPN'M relate to the area of ΔPQN if N' is the point two thirds of the way on the segment QM from Q to M?*
(c) *What is the relation between the area of ΔPNN' and the area of* ΔPQN?

19. In Exercise 18 suppose that we select N on the segment QM at a distance from Q equal to one-fourth of d(Q, M). What can you then say about the areas of ΔPQN and ΔPNM? What if this distance was one-fifth?

20. Prove that if the diagonals of a quadrilateral are perpendicular, then the area of the quadrilateral may be found by taking one-half the product of the lengths of the diagonals. (Draw a picture, and label all lengths clearly.)

D. PYTHAGORAS THEOREM

Let's build up squares on the sides of a right triangle. Pythagoras Theorem then claims that the sum of (the areas of) two small squares equals (the area of) the big one.

Theorem!
Pythagoras Theorem!
5.2. Let ΔXYZ be a right triangle with legs of lengths and y , and hypotenuse of length z. Then

$$x^2 + y^2 = z^2$$

See figure below for more detail explanation.

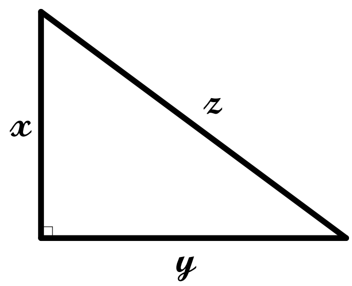


Figure 5.28. Triangle of Pythagoras Theorem.

In algebraic terms, $a^2 + b^2 = c^2$ where c is the hypotenuse while a and b are the legs of the triangle. The theorem is of fundamental importance in the Euclidean Geometry where it serves as a basis for the definition of distance between two points. It's so basic and well known that, we believe, anyone who took geometry classes in high school couldn't fail to remember it long after other math notions got thoroughly forgotten

There are so many proofs about the theorem; here we explain some proof to you.

Proof #1!
For general proofing, it is ordinary that to proof Pythagoras theorem, we only show that the sum of area two medium square equals to the area of the biggest square as shown below. In other word we write $c^2 = a^2 + b^2$.

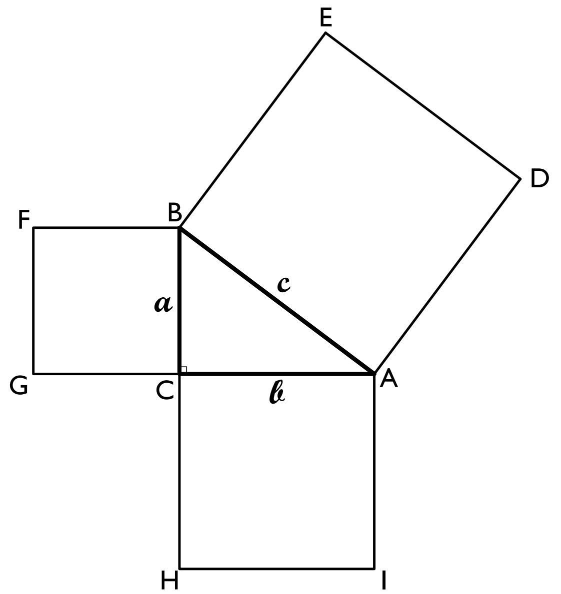


Figure 5.29. Triangle of Pythagoras Theorem.

To proof, please draw a line from vertex C perpendicular to line c.

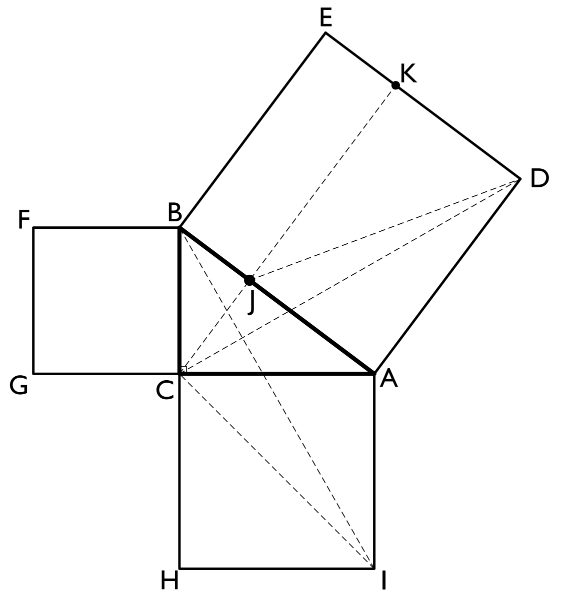


Figure 5.30. Proofing of Pythagoras Theorem. #1 way.

We divide square ABED into two parts.

We assume that $c^2 = A_{ABED}$ is held from $A_{BJKE} + A_{AJKD}$.
We will proof that $A_{BJKE} = a^2$ and $A_{AJKD} = b^2$.

Case 1 : we proof that $A_{AJKD} = b^2$.

Obvious $A_{\Delta AJD} = A_{\Delta ACD}$ *(why?)*.
Obvious $A_{\Delta AIC} = A_{\Delta AIB}$ *(why?)*.

Look at ΔAIB and ΔACD.

We have fact that
(1) $m\angle IAC = 90° = m\angle JAD$,
(2) $m\angle IAB = m\angle IAC + m\angle CAB = 90° + m\angle CAB$, and
(3) $m\angle DAC = m\angle DAJ + m\angle BAC = 90° + m\angle CAB$.
From (1) until (3) we get $m\angle IAB = m\angle DAC$.

Obvious $|AB| = |AD| = c$,
$m\angle IAB = m\angle DAC$, because of fact above, and
$|AI| = |AC| = b$.

We conclude that $\Delta AIB \cong \Delta ACD$ ……… 1∗

The direct consequence is
$A_{\Delta AIB} = A_{\Delta ACD}$
$\Leftrightarrow A_{\Delta AIC} = A_{\Delta AJD}$
$\Leftrightarrow A_{AJKD} = A_{ACHI}$.

This prove that $A_{AJKD} = b^2$.

Case 2 : we proof that $A_{BJKE} = a^2$.

By same way, you may find that $A_{BJKE} = a^2$.
We leave this proof as exercise to the reader.

From case 1 and case 2, we conclude that $A_{ABED} = A_{BJKE} + A_{AJKD}$ $\Leftrightarrow c^2 = a^2 + b^2$.
These proof the theorem. □

Proof #2!
See figure below!

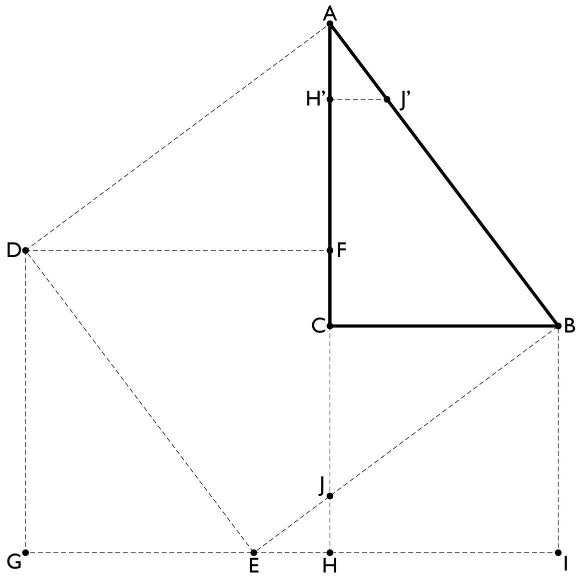


Figure 5.31. Proofing of Pythagoras Theorem. #2 ways.

To proof Pythagoras theorem in another way, we use figure 5.31 to get more detail information. We just show that A_{ABED} can be divided into A_{DFHG} and A_{BCHI}.

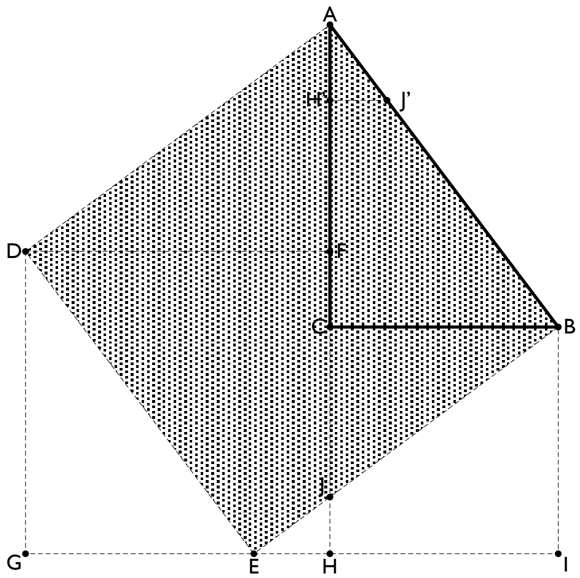


Figure 5.32. Proofing of Pythagoras Theorem. STEP #1.

Shaded region on figure above show us the area of square DABE. We make ΔAFD from ΔABC rotated 90°. So, we sure that is $\Delta AFD \cong \Delta ABC$. So that the area of square DABE equals c^2.
We will show that $A_{DABE} = A_{DFHG} + A_{BCHI}$ that is $c^2 = b^2 + a^2$

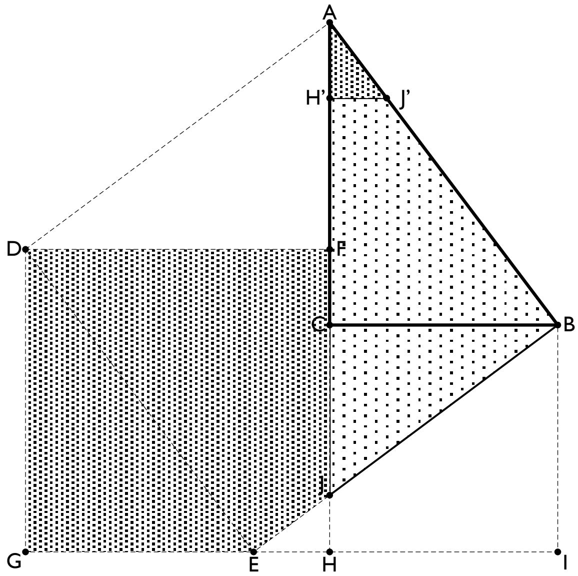


Figure 5.33. Proofing of Pythagoras Theorem. STEP #2.

Second step, we move ∆ADF to ∆DEG. This step is legal because ∆ADF ≅ ∆DEG *(why?)*. We draw different shading to give information to next step.

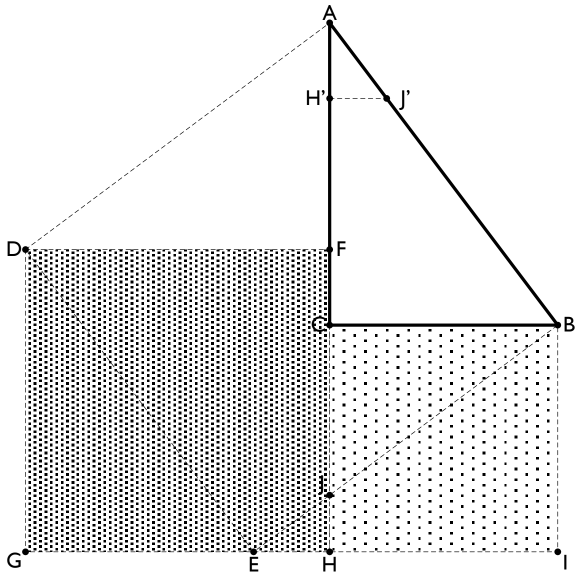


Figure 5.34. Proofing of Pythagoras Theorem. STEP #3.

Now, we have moved all possible congruent regions to proof Pythagoras Theorem. We have show you that $A_{DABE} = A_{DFHG} + A_{BCHI}$ that is $c^2 = b^2 + a^2$.
These prove the theorem. □

Application of Pythagoras Theorem!

We think that is important to give you an example how Pythagoras theorem is used. Here we give you a problem related to Pythagoras theorem.

Find the length of AG from 4 cm side length cube given below!

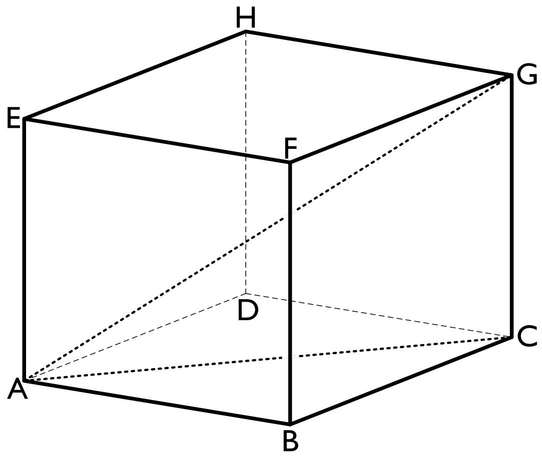


Figure 5.35. Cube on application of Phytagoras Theorem problem.

Obvious $|AG|^2 = |AC|^2 + |CG|^2 = (|AB|^2 + |BC|^2) + |CG|^2$
$= (4^2 + 4^2) + 4^2 = 16 + 16 + 16 = 48$

Obvious $|AG| = \sqrt{48} = \sqrt{16 . 3} = 4\sqrt{3}$

We get $|AG| = 4\sqrt{3}$ cm.

E. EXERCISE CHAPTER 5 #2

1. Look at figure 5.36 below! Find the length of the diagonal of a rectangular solid whose sides a, b, and c have lengths:
   (a) 3, 4, 5 (b) 1, 2, 4 (c) 1, 3, 4 (d) a, b, c (e) ra, rb, rc

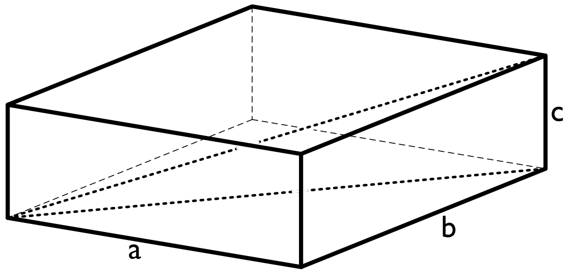


Figure 5.36.

2. A TV tower is to be anchored by wires from a point 30 m above the ground to stakes set 25 m from the base of the tower. How long does each wire have to be?

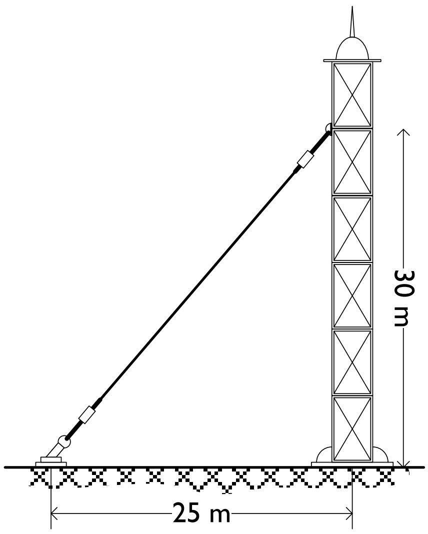


Figure 5.37.

3. How far is it from second base to home plate? (Note: A baseball "diamond" is really a square, with side's length 90 feet.)
4. A square has area 144 sq. cm. What is the length of its diagonal?
5. One leg of a right triangle is twice the length of the other leg. The area of the triangle is 72 sq. cm. How long is the hypotenuse?

6. In the figure below, right triangle PQA_1 has two legs of length 1.
   (a) How long is hypotenuse PA_1?
   (b) How long is segment PA_2?
   (c) How long is segment PA_3?
   (d) How long is the "n-th" segment, PA_n?

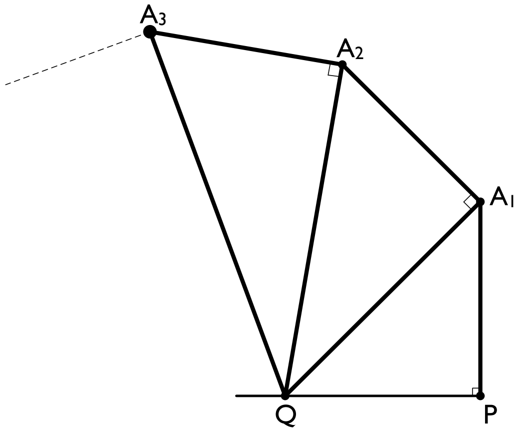


Figure 5.38.

7. (a) A ship travels 6 km due South, 5 km due East, and then 4 km due South. How far is it from its starting point? (The answer is NOT 15 km!) You may assume that the path of the ship is as shown on the diagram, and lies in a plane.

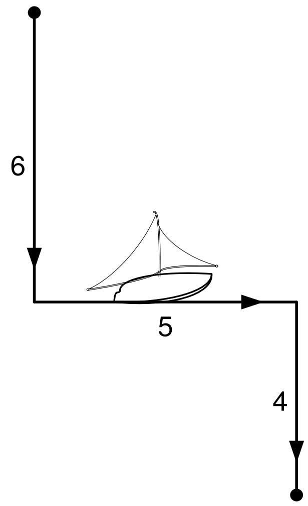


Figure 5.39.

(b) If actually the ship starts at the North Pole, on earth, what would the actual distance be?

8. (a) A right triangle ABC has legs of lengths 3 and 4. What is the length of the hypotenuse?
   (b) Suppose we double the lengths of the legs, so that they are 6 and 8. How long is the hypotenuse in this case?
   (c) Now triple the original lengths of the legs. How long is the hypotenuse in this case?
   (d) Suppose we multiply the lengths of the original legs by a factor of e (e a number greater than 0), so that they have lengths 3e and 4e. What is the length of the hypotenuse?
   (e) Prove your conclusion in part (d).
   (f) In your head, compute the lengths of the hypotenuses of the following right triangles whose legs are given:
   (i) 300 and 400 (ii) 18 and 24 (iii) 27 and 36

9. In Problem 8, you found a number of Pythagorean triples, which are sets of three integers a, b, and e which satisfy

$$a^2 + b^2 = c^2$$

Can you find other triples which are NOT multiples of the ones in Problem 8? For example, 5, 12, 13 is such a triple ($5^2 + 12^2 = 13^2$). Can you develop a calculation method which will generate Pythagorean triples?

[Hint: Let x = $\frac{a}{c}$ and y = $\frac{b}{c}$ so $x^2 + y^2 = 1$. Use the formulas

$$x = \frac{1 - t^2}{1 + t^2} \; and \; y = \frac{2t}{1 + t^2}$$

Substitute values for t, like t = 2, t = 3, t = 4, t = 7, or whatever.]

10. In each of the following right triangles, find x (think of Pythagorean triples).

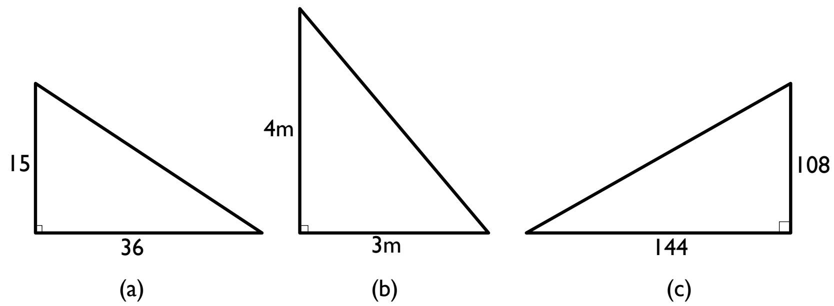


Figure 5.40.

11. The lengths of the legs of a right triangle are in the ratio 2: 3. If the area of the triangle is 27, how long is the hypotenuse?

12. One leg of a right triangle is 4/5 of the other. The area of the triangle is 320. Find the lengths of the legs.

13. A 20 m pole at one corner of a rectangular field is anchored by a wire stretched from the top of the pole to the opposite corner of the field as shown:

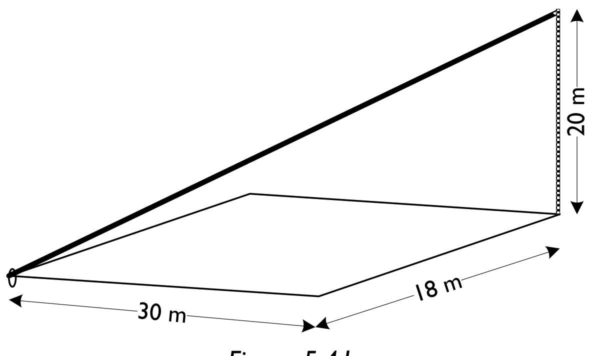


Figure 5.41.

What is the length of the wire?

14. A smaller square is created inside a larger square by connecting the midpoints of the sides of the larger square, as shown:

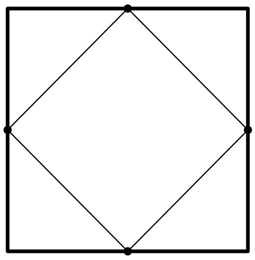


Figure 5.42.

What is the ratio of the area of the small square to the area of the big square? Proof it!

15. A baseball diamond is a square 90 feet on a side. If a fielder caught a fly on the first base line 30 feet beyond first base, how far would he have to throw to get the ball to third base?
16. A square has area 169 cm^2. What is the length of its diagonal?
17. Find the area of a square with a diagonal of length $8\sqrt{2}$ cm.
18. Find the area of a square whose diagonal has length 8.
19. The length and width of a TV screen must be in the ratio 4: 3, by FCC regulation. If a company advertises a portable TV with a screen measuring 25 cm along the diagonal:
    (a) What is the viewing area in square centimeters?
    (b) If the diagonal measurement is doubled to 50 cm, does the viewing area double? If NO, how does it change?

20. In the figure, find |PQ|.

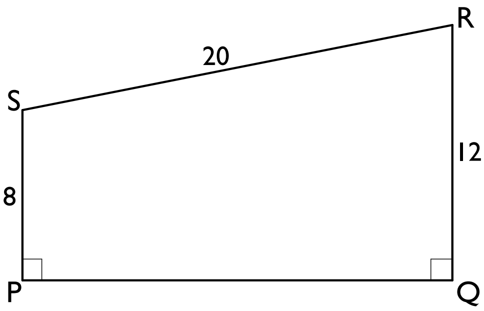


Figure 5.43.

21. Use Pythagoras to prove the following statement, which is an extension of the **RT postulate**:

    If a leg and hypotenuse of one right triangle have the same length as a leg and hypotenuse respectively of another right triangle, then the third sides of each triangle have the same length.

22. ABCD is a rectangle, and AQD is a right triangle, as shown in the figure. If |AB| = a, |BQ| = b, and |QC| = c, prove the following:
    (a) $|AD| = \sqrt{b^2 + 2a^2 + c^2}$
    (b) $a^2 = bc$ (This should be easy once you've shown part (a))

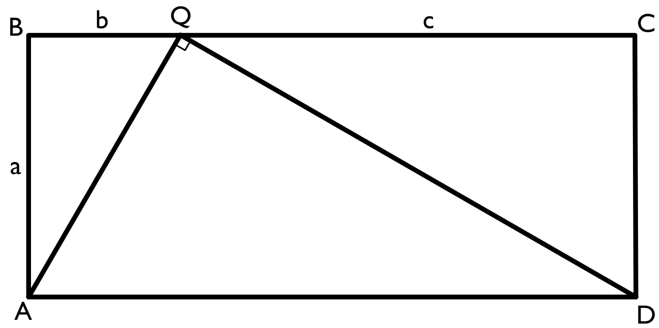


Figure 5.44.

23. In triangle ABC, |AC| = |CB| and CN is drawn perpendicular to AB. Prove that |AN| = |NB|.

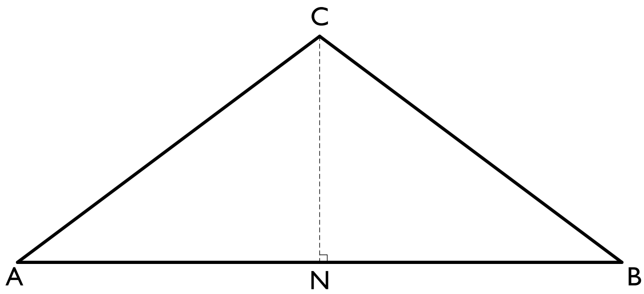


Figure 5.45.

24. Let ΔABC be a right triangle as shown on Figure below, with right angle C.

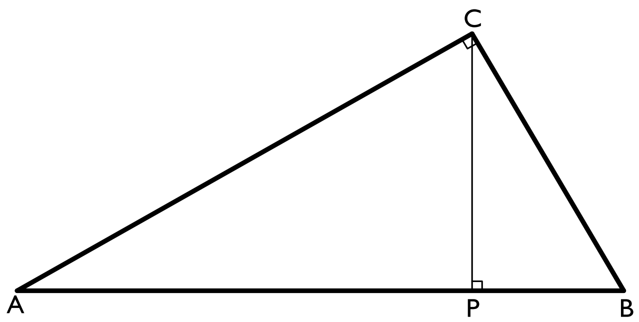


Figure 5.46.

Let P be the point on AB such that CP is perpendicular to AB. Prove:
(a) $|PC|^2 = |AP| \, . \, |PB|$
(b) $|AC|^2 = |AP| \, . \, |AB|$

25. Let PQ and XY be two parallel segments. Suppose that line L is perpendicular to these segments, and intersects the segments in their midpoint. (We shall study such lines more extensively later. The line L is called the perpendicular bisector of the segments.)

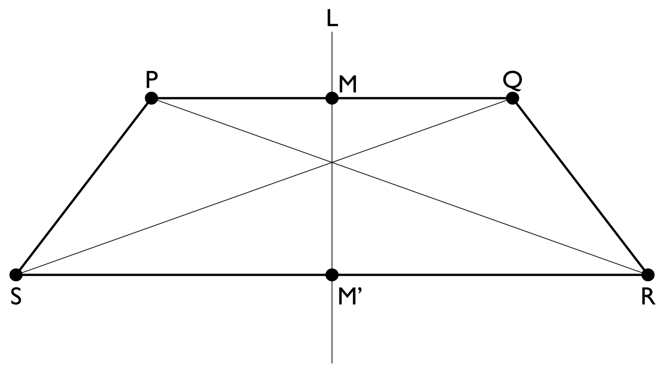


Figure 5.47.

It is given L ⊥ PQ and L ⊥ RS.
|PM| = |MQ| and |SM'| = |M'R|.

Prove:
(a) |PX| = |QY|
(b) |PY| = |QX|
[Hint: Use the lines perpendicular to PQ and XY passing through P and Q and intersecting X Y in points P' and Q' as on Figure below. Use various equalities of segments, and Pythagoras' theorem.]

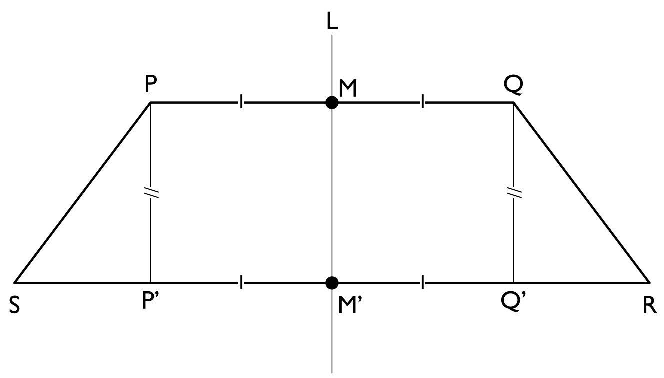


Figure 5.48.

Remark! Statements (a) and (b) can be interpreted by saying that if two points are reflected through a line, then the distance between the two points is the same as the distance between the reflected points. The two points might be P, X or they might be P, Y, depending on whether they lie on one side of the line L or on different sides of the line L. F or a more systematic study of reflections.

26. Prove that the length of the hypotenuse of a right triangle is ≥ the length of a leg.

27. The next five exercises are "data sufficiency" questions. In this type of question, you are asked to make a specific calculation (in this case the area of a particular square), using the data given in the question. If the given data is sufficient to determine the answer, you do so. If it is not sufficient, you answer "insufficient data". Each question is independent of the others.

    Three squares intersect as shown in Figure below. Find the area of △ADG for the following sets of data.

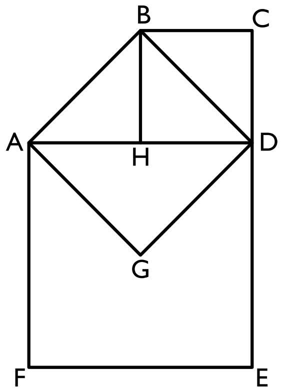


Figure 5.49.

(a) |AF| = 10
(b) |CE| = 18
(c) |BD| = $3\sqrt{2}$
(d) Area of square BCDH = 49
(e) Area of figure AGDEF = 27

Chapter 6

T H E N E X T P A R A L L E L L A W

Some subject material has explained at chapter 2, the locus of geometry objects. Now, we will discuss the parallel as higher level of parallel.

A. DIRECT PROPORTION OF LINES

If we talk about direct proportion it's like discuss about scale. We will talk about when the value of scale become larger, then in fact, the things in the scale become larger too.

Theorem!

6.1. If some parallel lines intersect same parts of any cross line, then the parallel lines will intersect to same part of another cross lines.

See figure below for more detail explanation.

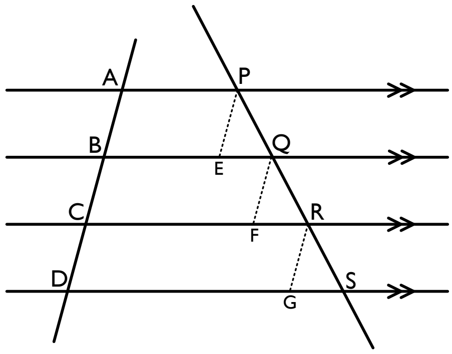


Figure 6.1. Condition of Line Direct Proportion

It is given four parallel lines, AP || BQ || CR ||/ DS.
It is known that AB = BC = CD
The theorem is just saying that:

$$PQ = QR = RS.$$

Proof!
From each P, Q, R, S, builds segment parallel to AD crossing the parallel line below. Now, we get 3 parallelograms ABEP, BCFQ, CDGR. The direct consequence is the length of AB = PE, BC = QF, CD = RG.

Obvious $m\angle PEQ = m\angle QFR = m\angle RGS$, (parallelogram property.),
$m\angle PQE = m\angle QRF = m\angle RSG$, (axiom 2.2.), and
$|PE| = |QF| = |RG|$.
We conclude that $\Delta PQE \cong \Delta QFR \cong \Delta RGS$ because of **SAA** rule.
The direct consequences are $|PQ| = |QR| = |RS|$. □

Theorem!

6.2. If some parallel lines intersects same parts of any intersect line, then the parallel lines will intersect to same part of another intersect lines.

See figure below for more detail explanation.

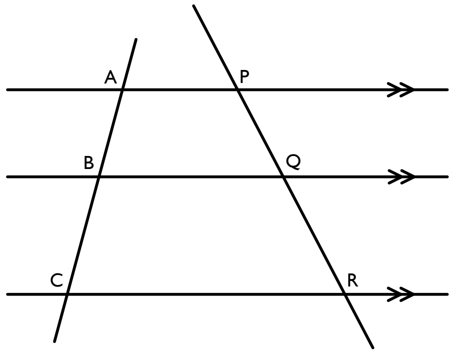


Figure 6.2. Condition to Th. 5.2

Analyze.
It is given three parallel lines AP || BQ || CR. See figure 5.2.
Prove that AB : BC = PQ : QR.

Proof!
Suppose that AB : BC = 4 : 7.
So, we can divide AB into 4 parts by making 4 parallel lines which are parallel to AP and also with BC. See figure 6.3 below.

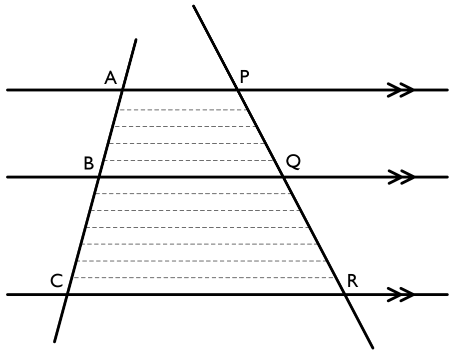


Figure 6.3. Explanation of Th. 5.2

From theorem 6.1 we knew that the parallel line divided AB and PQ in similar parts. And its also done in BC and QR.
So that we get PQ : QR is also 4 : 7.
This proves the theorem. □

Theorem!

6.3. A line which is parallel to one side of a triangle, is dividing another two sides with similar comparison.

See figure below for more detail explanation.

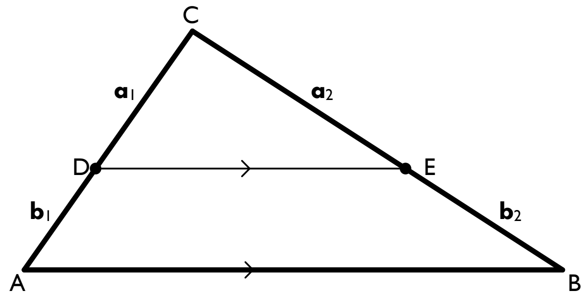


Figure 6.4. Explanation of Th. 6.3

Let ABC is a triangle, DE ∥ AB, and
a$_1$ = |CD|, **a**$_2$ = |CE|, **b**$_1$ = |DA|, **b**$_2$ = |EB|.

The theorem is just explain us that,

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}$$

Proof!
Draw a line through C parallel to DE and AB. *See figure 6.5*

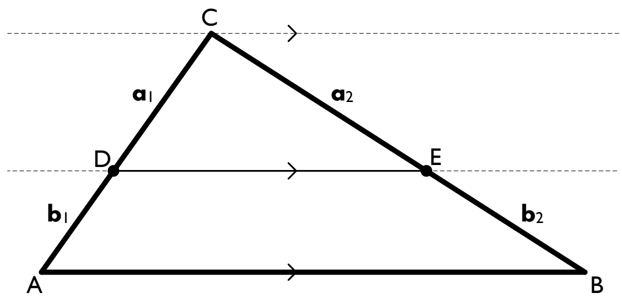


Figure 6.5. Explanation of Th. 6.3

Theorem 6.2 said that $\frac{CD}{DA} = \frac{CE}{EB}$ or somewhat we say it $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.
These proof this theorem. □

By using the properties of direct proportion,

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}$$

The consequences of it are:

1. $\frac{(a_1+b_1)}{(a_2+b_2)} = \frac{a_1}{a_2}$ or $\frac{(a_1+b_1)}{(a_2+b_2)} = \frac{b_1}{b_2}$

2. $\frac{(1)}{(2)} = \frac{a_1}{a_2} = \frac{CA}{CB} = \frac{CD}{CE}$

3. $\frac{(1)}{(2)} = \frac{b_1}{b_2} = \frac{CA}{CB} = \frac{DB}{EA}$

Theorem!

6.4. A line which is parallel to one side of a triangle, will build a triangle. The corresponding side of new triangle has similar proportion to the latest one.

See figure below for more detail explanation.

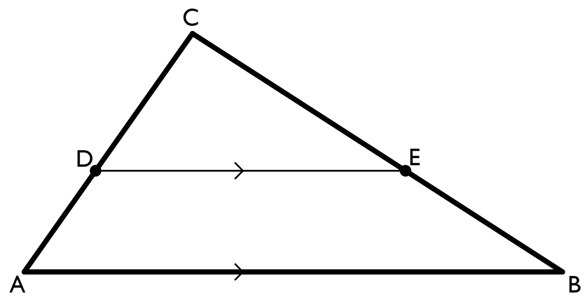


Figure 6.6. Explanation of Th. 5.4

Let ABC is a triangle and DE || AB.
The theorem explain us that,

$$\frac{CD}{CA} = \frac{CE}{CB} = \frac{DE}{AB}$$

Proof!
Build a segment EF, F on AB, such that EF || AC. See figure 6.7 below.

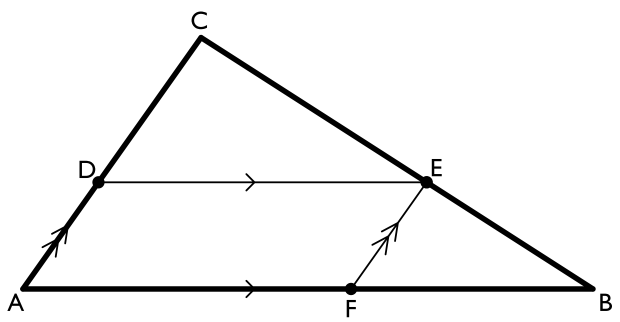


Figure 6.7. Explanation of Th. 5.4

Look at ΔCDE and ΔCAB.
From the consequences of theorem 6.3 (2) above, we conclude that:

$$\frac{CA}{CB} = \frac{CD}{CE}$$

$$\Leftrightarrow \qquad CA.CE = CD.CB$$

$$\Leftrightarrow \qquad \frac{CE}{CB} = \frac{CD}{CA} \qquad \ldots\ldots\ldots\ 1*$$

Look at △BFE and △BAC.
From the consequences of theorem 6.3 (3) above, we conclude that:

$$\frac{BC}{BA} = \frac{EC}{FA}$$

$$\Leftrightarrow \qquad BC.FA = BA.EC$$

$$\Leftrightarrow \qquad CB.DE = AB.CE$$

$$\Leftrightarrow \qquad \frac{CE}{CB} = \frac{DE}{AB} \qquad \ldots\ldots\ldots\ 2*$$

From 1* and 2*, we conclude that $\frac{CE}{CB} = \frac{CD}{CA} = \frac{DE}{AB}$.
This proves the theorem. □

Theorem!

6.5. If a line cross side AC and BC of a triangle in D and E, such that $\frac{CD}{CA} = \frac{CE}{CB}$, then the line DE is parallel to AB.

See figure below for more detail explanation.

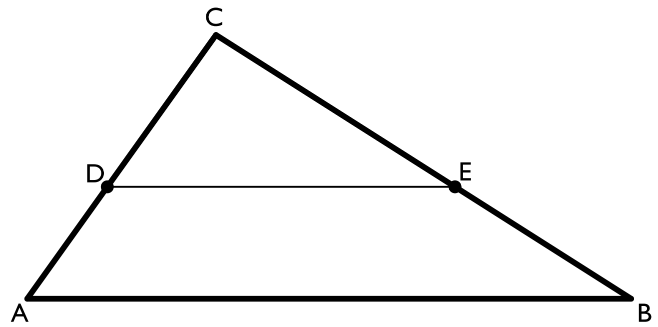


Figure 6.8. Explanation of Th. 5.5

Let ABC is a triangle and DE ∥ AB.
The theorem is just explain us that,

$$\left(\frac{CD}{CA} = \frac{CE}{CB}\right) \Rightarrow DE \parallel AB.$$

Proof!
We have $\left(\frac{CD}{CA} = \frac{CE}{CB}\right)$. We will prove that $DE \parallel AB$.

Suppose that $DE \nparallel AB$.

From **axiom 1.3** there is only one line through point D which is parallel to AB. Please give label it by line DF, F on BC. That hold DF || AB. *See figure 6.9 below.*

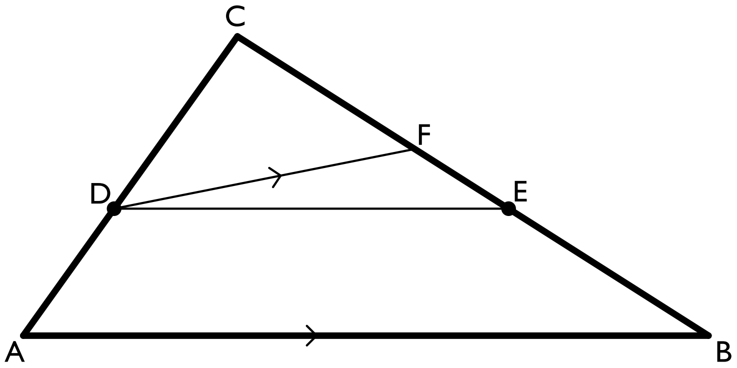


Figure 6.9. Explanation of Th. 5.5

From th. 5.4, It consequence $\left(\frac{CD}{CA} = \frac{CF}{CB}\right)$. Because we have $\left(\frac{CD}{CA} = \frac{CE}{CB}\right)$, the situation become $\frac{CD}{CA} = \frac{CF}{CB} = \frac{CE}{CB}$. From the fact that $\frac{CF}{CB} = \frac{CE}{CB}$ it seem like CF = CE, then we get F = E. It hold a contradict because we knew that F ≠ E.
So the supposition is false. The true one is DE || AB.
This proves the theorem. □

B. DRAWING ALGEBRAIC PLANAR

In this sub chapter, we will explain two constructions. These constructions based on the theorems before. We let the reader to explore another construction itself.

1. Construction 6.1.

Dividing a line into some similar parts!

In solid geometry, we often use this construction to divide a line suitable to its direct proportion. The direct proportion in solid means the ratio of orthogonal of projection to the real. We may use this construction to make a direct proportion of the line.

Question:
It is given a segment AB below. Divide AB into 3 similar parts!

Figure 6.10. Question on const. 6.1

Answer.
This construction use theorem 6.1 as the basic theorem!
From point A, make a segment. The principal to answer this question is just making parallel lines from a helped line.

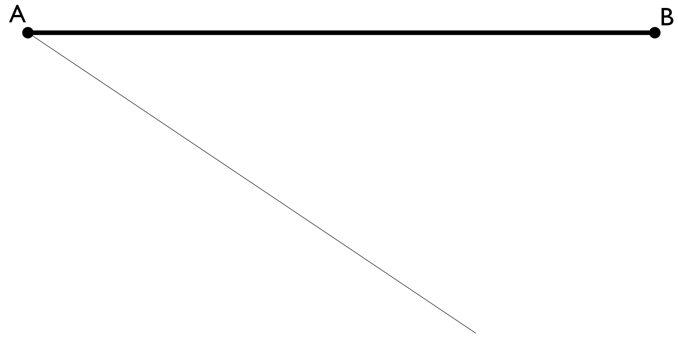


Figure 6.11. Making a helped line

Figure 6.11 is like the condition of Th. 6.1 if the both cross line are extended until meet at one point. The next principal is just making parallel lines intersecting both lines. But before that we have to make 3 similar parts using the helped line.

Make an arc from point A until the arc cross the helped line, label it with P'. By same length radius, do same step, just change the center of arc with P' and find the point Q'. Do same step until you find R'. See figure below to simplify the concept. You get d(AP') = d(P'Q') = d(Q'R').

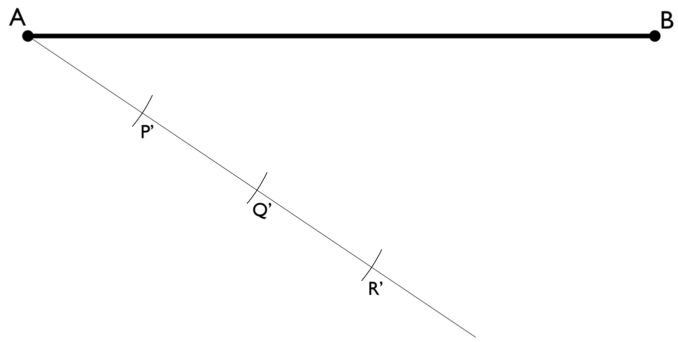


Figure 6.12. Making 3 similar parts on helped line.

Next step is connecting R' to B! By using two triangle rulers, make parallel line with BR' through Q' and cross AB at Q, through P' and cross AB at P.

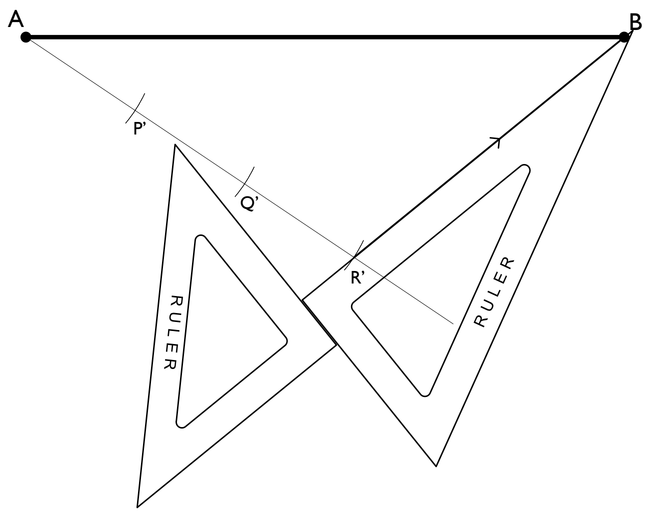


Figure 6.13. Making 3 similar parts on AB. STEP #1

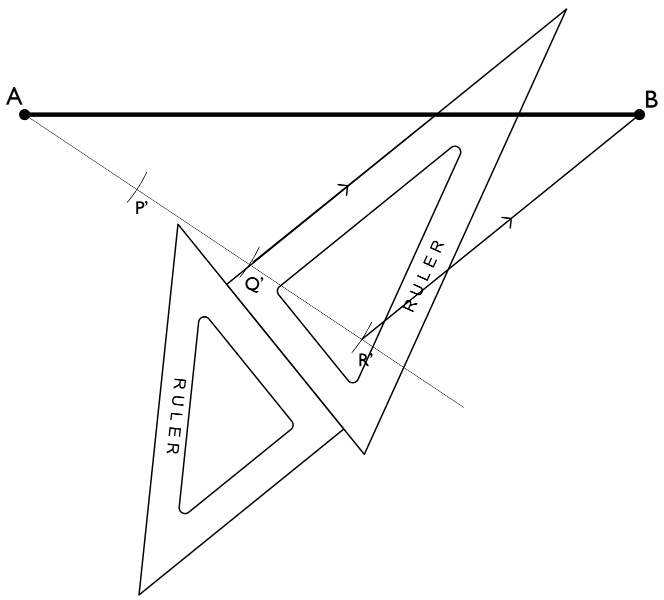


Figure 6.14. Making 3 similar parts on AB. STEP #2

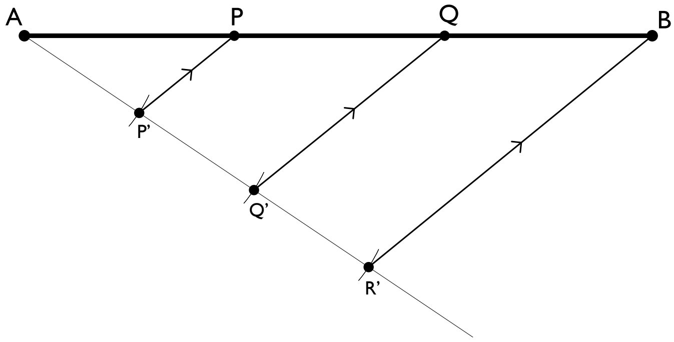


Figure 6.15. Making 3 similar parts on AB. The Result

From theorem 6.1, we get that d(A,P) = d(P,Q) = d(Q,B). This solves the problem on construction 6.1.

2. Construction 6.2.

Making a projection ratio between orthogonal and the real!

We will construct a new topic in here, because this material is important for higher level geometry. We introduce the word projection ratio and orthogonal. Projection ratio is a proportion between orthogonal to real. Orthogonal line or plane is a geometry subject that is perpendicular to drawing plane. *See figure 5.16 below.*

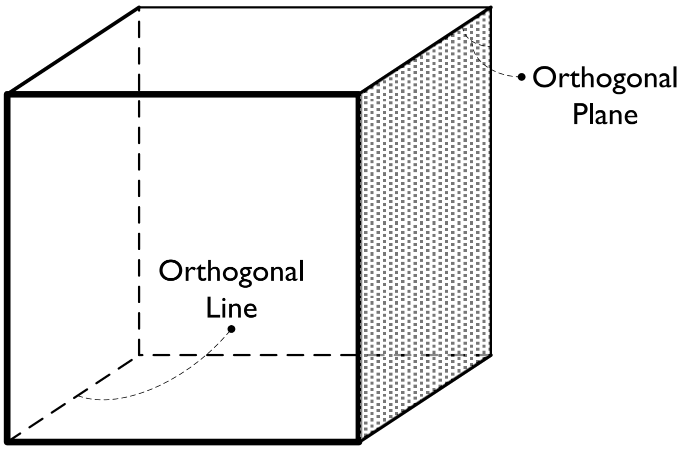


Figure 6.16. Introducing Orthogonal Line and Orthogonal Plane

Now, it is the construction.

Question!
Draw an orthogonal line PQ, which has projection ratio 2 : 3 to AB, (PQ = AB) and has subsided angle 60°, through point D, such that PD : DQ = 2 : 1.

Figure 6.17. Condition of construction 6.2

Answer.
We do like construction 6.1, which is divide line AB to 3 similar parts. Thus, we know that 2 of 3 parts is the length of PQ.

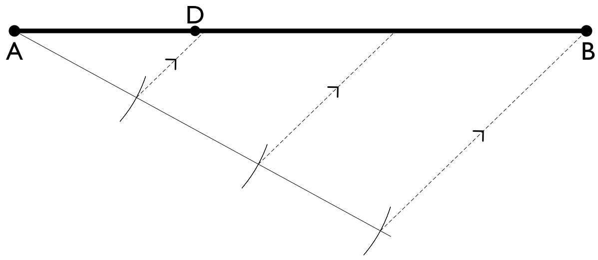


Figure 6.18. Making projection ratio PQ, which has ratio 2 : 3 of AB. STEP #1

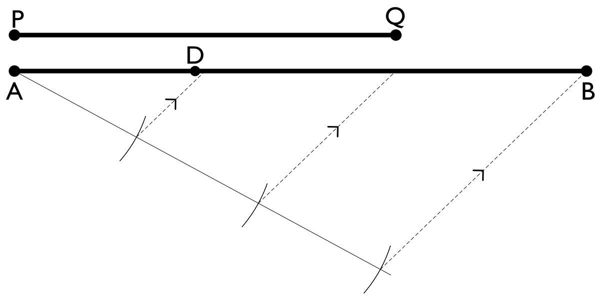


Figure 6.19. Making projection ratio PQ, which has ratio 2 : 3 of AB. STEP #2

Our project now is dividing PQ into 3 similar parts. Thus, we will get point D as cross point of PQ and AB.

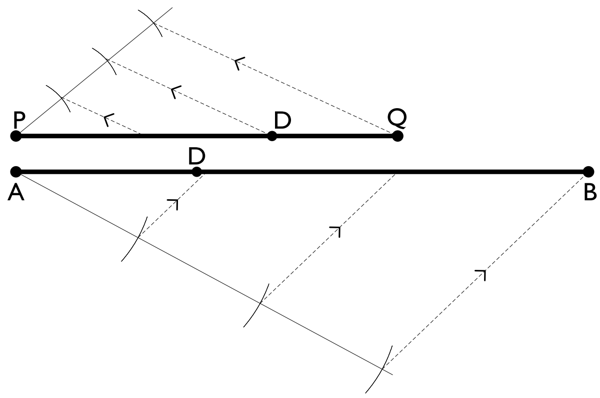


Figure 6.20. Making projection ratio PQ, which has ratio 2 : 3 of AB. STEP #3

Next step is just making 60° angle from point D as angle point.

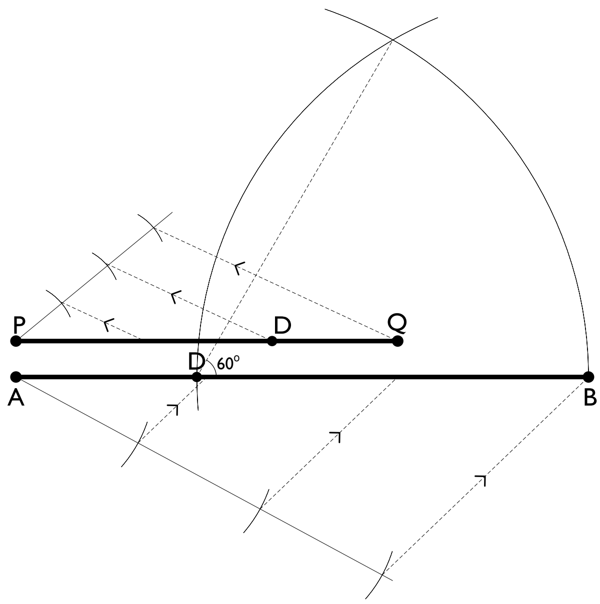


Figure 6.21. Making subsided angle 60°.

And the final step is coinciding PQ to helped line of 60° and both of point D.

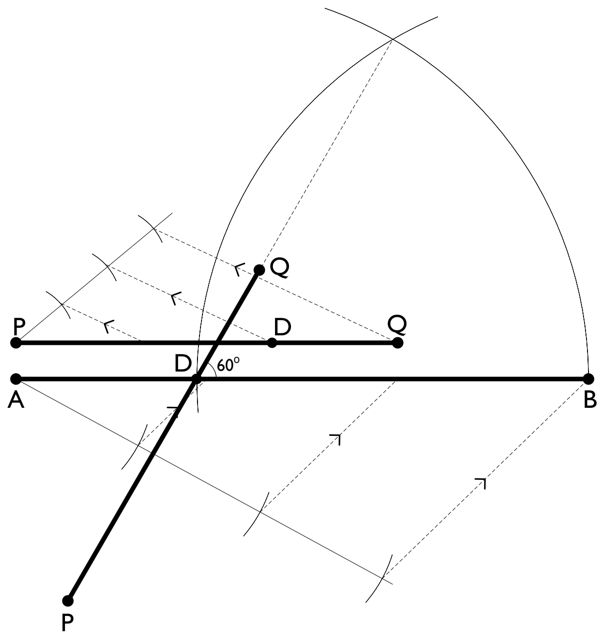


Figure 6.22. PQ has on its position.

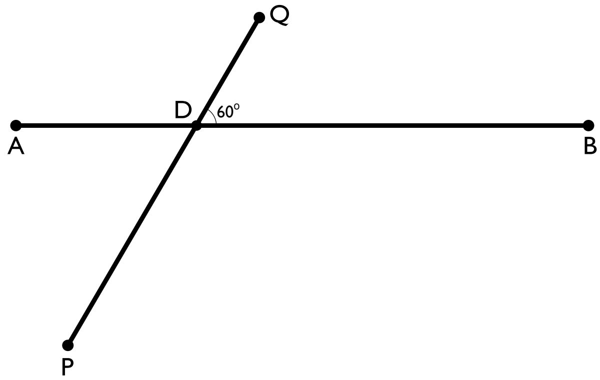


Figure 6.23. The final result of the construction.

Now, we have solved the problem on question above.

3. Construction 6.3.
Drawing a fourth line from three another line given!

We introduce the term comparer.
Let there are algebraic form $\frac{a}{b} = \frac{c}{x}$. We let x as the fourth compared to a, b, and c.
From $\frac{a}{b} = \frac{c}{x}$ we get $ax = bc$ or $x = \frac{bc}{a}$.

In geometry, if a, b, and c are given lines, then x could be drawn by theorem 6.3. See figure below to get more detail information above.

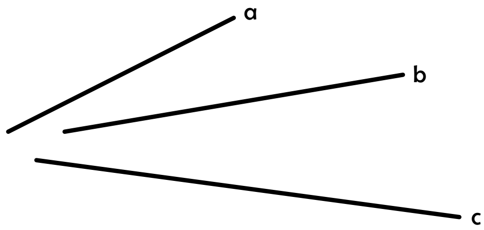


Figure 6.24. Condition of information above.

The problem is find the length of x, a fourth line, such that $x = \frac{bc}{a}$.

Solution!
First step to make solution is making an angle P.

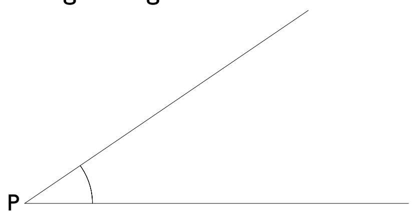


Figure 6.25. Any angle P as helped angle.

Then put line **a** to one leg of angle P, together with line **b**. Let there are point A such that |PA| = **a** and point B such that |AB| = **b**.

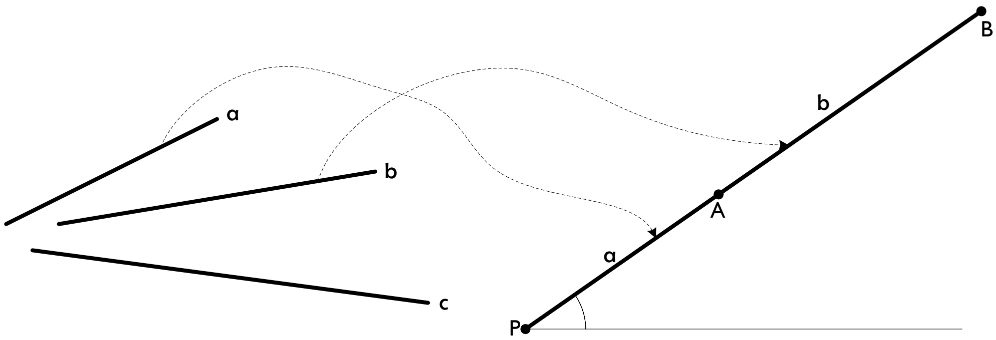


Figure 6.26. Draw a fourth line. STEP #1.

Do next step with line c, and we will get the result as shown below.

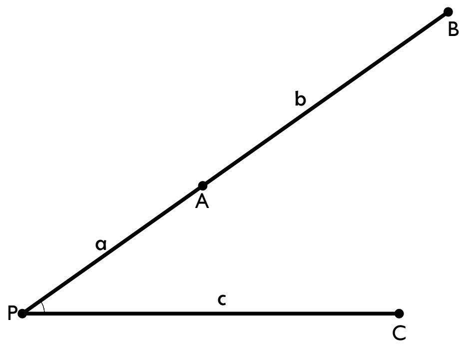


Figure 6.27. Draw a fourth line. STEP #2.

Connect A and C, after that please extend PC!

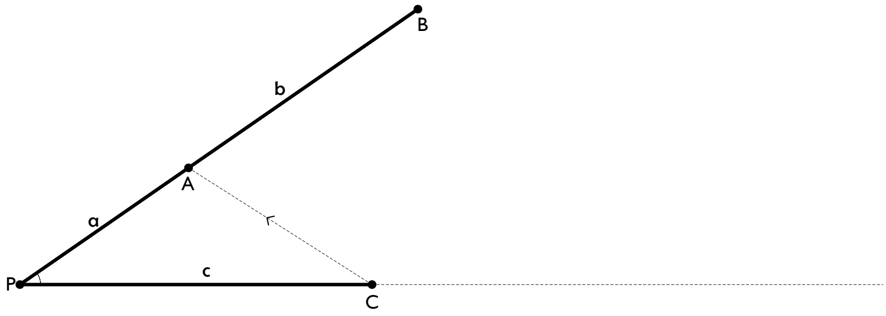


Figure 6.28. Draw a fourth line. STEP #3.

Next step is drawing a line from B parallel with AC until cross the extension of PC, label it with D. Thus, we get the length of x is equal to the length of CD.

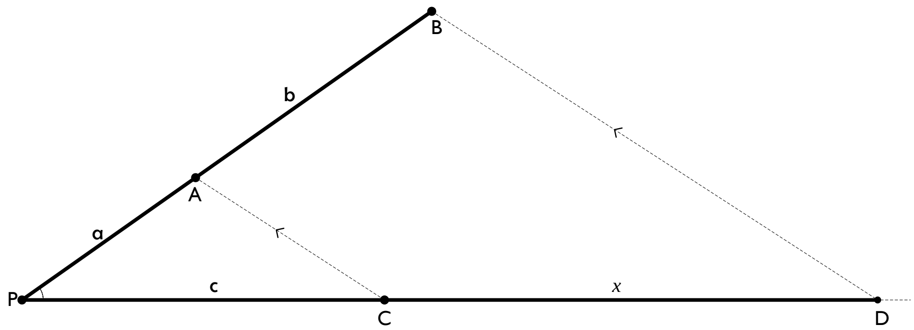


Figure 6.29. Draw a fourth line. The final result.

Now, we have get the length of x such that $x = \frac{bc}{a}$.
Thus, we have solved the problem.

This is finishing the construction on chapter 6.

C. DILATION

We have learned application of dilation on similarity and congruency on chapter 3. We knew that congruency held if there were two planar and each other has dilation by factor 1 to another planar. Now our discussion will be more detail by dilation to real number such that the dilate factor is not only 1 but also negative and fraction.

Look at the figure below.

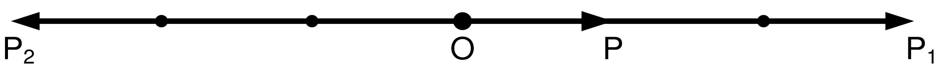


Figure 6.30. Some position of the shadow of P.

In figure 6.30, P_1 is on the right side of ray OP. Because of the length OP_1 is three times of |OP|, we say that **OP is dilating by dilate factor 3 with O as the center** become OP_1. Almost similar to P_1, P_2 is on the left side of P. Because of the length OP_2 is three times of |OP|, we say that OP is dilating by dilate factor -3 with O as the center become OP_2. We may write it:

$$\left.\begin{aligned} OP_1 &= 3 \times OP \\ &\text{and} \\ OP_2 &= -3 \times OP \end{aligned}\right\} \ldots\ldots\ldots\ 3*$$

Give attention to 3*

1. The number 3 or –3 is called **dilate factor**,
2. The point O or center of dilation is called **sentrum,**
3. The point P is called **origin**, and
4. P_1 and P_2 are called the **shadows**.

Next topics are about dilation related to parallel law.

Theorem!

6.6. Let AB line. The shadow of line AB is line A_1B_1 which is parallel to AB. The length of A_1B_1 is r times to length of AB; r is dilate factor of dilation AB to A_1B_1.

See figure below for more detail explanation.

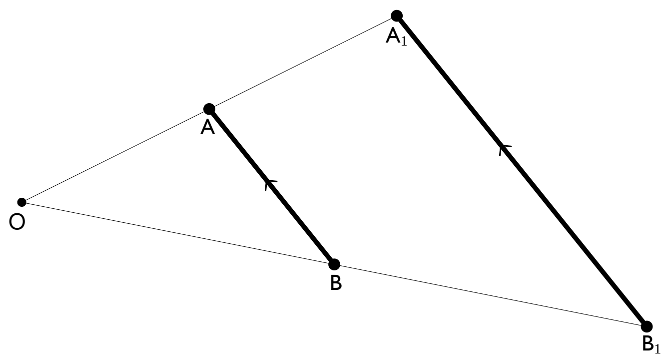


Figure 6.31. Condition of th. 6.6.

It is given line AB, $d(A_1O) = r \, . \, d(AO)$, and $d(B_1O) = r \, . \, d(BO)$.
The theorem is just said that AB || A_1B_1, $|A_1B_1| = r \, . \, |AB|$.

Proof!
We will proof that AB || A_1B_1.
Look at figure 6.31. Let there are ΔOA_1B_1.

Obvious $d(A_1O) = r \, . \, d(AO) \Leftrightarrow r = \dfrac{d(A_1O)}{d(AO)}$ and

$d(B_1O) = r \, . \, d(BO) \Leftrightarrow r = \dfrac{d(B_1O)}{d(BO)}$.

From theorem 6.5, we conclude that AB || A_1B_1.

We will proof that $|A_1B_1| = r \, . \, |AB|$.
Look at figure 6.31. Let there are ΔOA_1B_1.
We get AB || A_1B_1.

Obvious $\dfrac{d(A_1O)}{d(AO)} = \dfrac{d(B_1O)}{d(BO)} = r$.

From theorem 6.4 we have the corresponding side has similar proportion. We get that:

$$\frac{d(A_1O)}{d(AO)} = \frac{d(B_1O)}{d(BO)} = \frac{d(A_1B_1)}{d(AB)} = r$$

We know that $\frac{d(A_1B_1)}{d(AB)} = r$ or d(A_1B_1) = r . d(AB). In other written, we write $|A_1B_1| = r \cdot |AB|$.
This proves the theorem. □

Note:
Here we give an example if a line dilate by negative dilate factor.

Let AB any line. See figure 6.32 below. Dilate AB to A'B' by dilate factor $-\frac{1}{2}$.

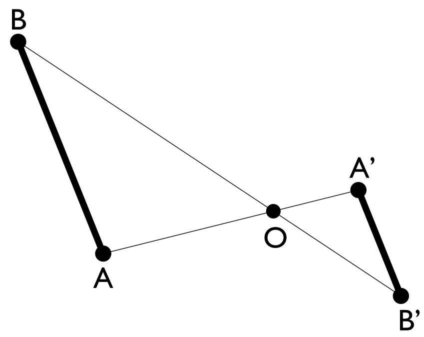


Figure 6.1. Dilation AB to A'B'.

From the figure above AB and A'B' has different ray. It is caused from negative dilate factor. The sentrum of dilation is placed on between the line and the shadow, and then as consequence of dilation, the length of A'O is a half of the length of AO because of dilate factor $-\frac{1}{2}$.

Theorem

6.7. Let any ∠A. The shadows of dilation ∠A is ∠A', m∠A' = m∠A, and the legs of ∠A' is parallel with the legs of ∠A.

See figure below for more detail explanation.

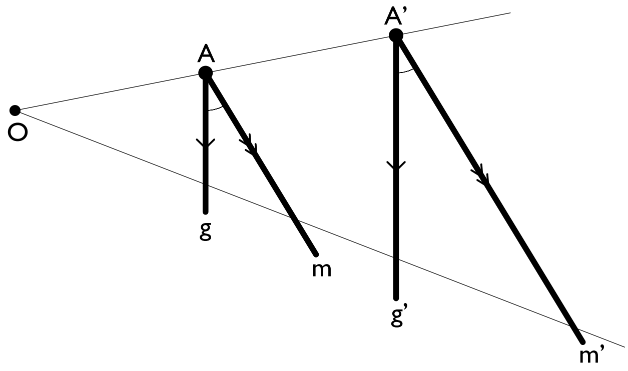


Figure 6.33. Condition of th. 6.7.

Let any ∠A. g and m are both of its legs.
The theorem is just say that

1. m∠A' = m∠A,
2. g ‖ g'
3. m ‖ m'

Proof!
Usually we proof number 1 first and continue to next number. But in this case, maybe easier if we proof number 2 and 3 first.

Now, we will proof g ‖ g' and m ‖ m'.
We divide case of two legs one by one. Let there is line g. Line g is dilating to line g'. From theorem 6.6 we conclude that g ‖ g'. This caused m ‖ m'.

To proof that m∠A' = m∠A, look at the figure below.

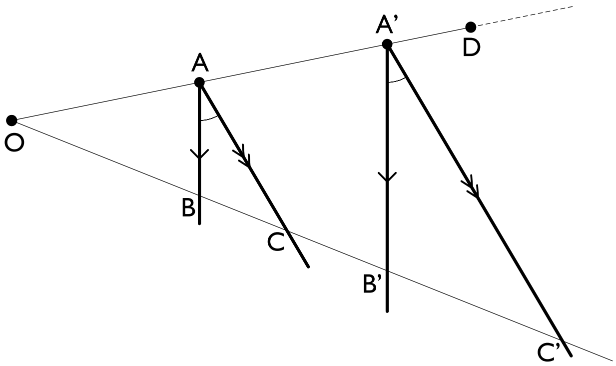


Figure 6.34. Construction of parallel line.

Because AB ‖ A'B' and because of axiom 2.2 we conclude that m∠BAA' = m∠B'A'D.
Because AC ‖ A'C' and because of axiom 2.2 we conclude that m∠CAA' = m∠C'A'D.
Obvious m∠BAA' – m∠CAA' = m∠B'A'D – m∠C'A'D
⇔ m∠BAC = m∠B'A'C'
⇔ m∠A = m∠A'.

Now we have proved that g ‖ g' , m ‖ m' , and also m∠A = m∠A'.
This proof the theorem. □

Theorem!

6.8. If a polygon dilates by dilate factor r or –r, and then the shadow is also polygon. The sides length of the shadow are r times the length of corresponding side. The angles size of the shadow is same with corresponding angle of the polygon.

Theorem!

6.9. On the corresponding planar (one planar is product of dilation), all corresponding angles have same size and lines are parallel to its corresponding. Corresponding planar will held bunch direct proportion.

See figure below for more detail explanation.

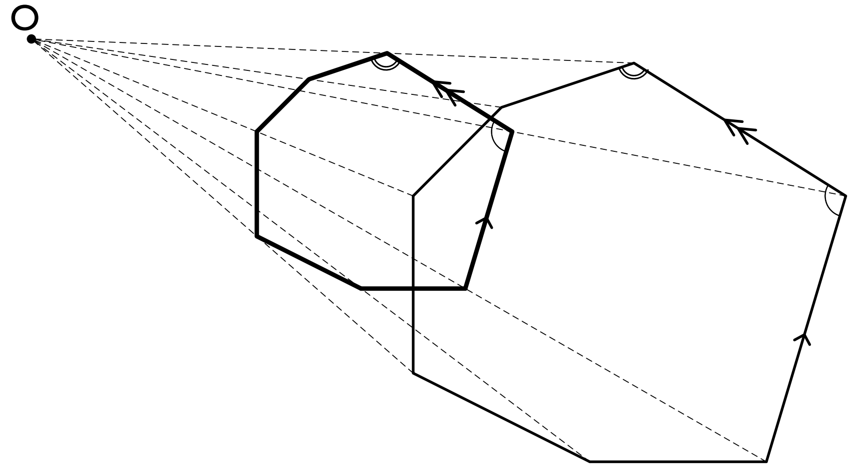


Figure 6.35. Dilation of a polygon.

On a triangle, dilation to *r* dilate factor is not only dilating its side. It is also dilating the altitude, its median, and also dilating its bisector to be *r* times the origin. See figure below to get more detail information.

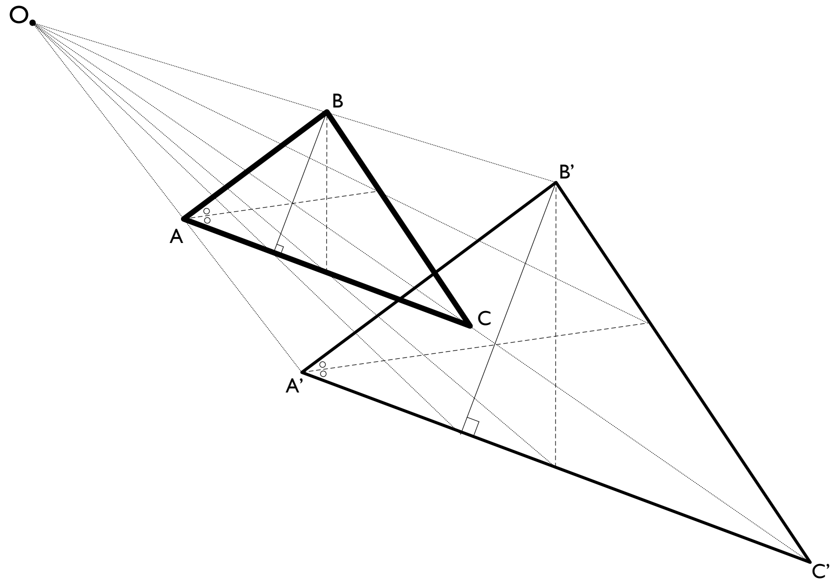


Figure 6.36. Dilation of a triangle was caused the altitude, the median, and the bisector also dilated.

From figure 6.36, we get d(A'B') = r . d(AB) ...... 4∗
and d(A'C') = r . d(AC) ...... 5∗

From 4∗ and 5∗ we get:

$$d(A'B') + d(A'C') = r . d(AB) + r . d(AC)$$
$$\Leftrightarrow d(A'B') + d(A'C') = r . (d(AB) + d(AC))$$
$$\Leftrightarrow r = \frac{d(A'B') + d(A'C')}{d(AB) + d(AC)} \quad .... 6*$$

from 6∗, we can write theorem below.

Theorem!

6.10. The sum or subtraction of the length of corresponding side has comparison to the origin. The value of comparison both equals the value of dilates factor r of dilation.

Proof!
We leave proof of theorem 6.8, 6.9, and 6.10 as exercises to the reader.

We give you an example how to draw a dilation of a planar.

Construction 6.3!
Drawing the shadow of dilation from a planar!

Question!
Let a triangle ABC below. Let O as the sentrum. Dilate ∆ABC by **(O, 3), that is dilate by sentrum O and dilate factor 3**.

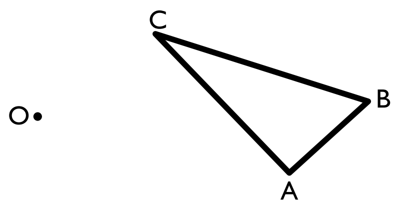


Figure 6.37. Dilation of a triangle by (O,3).

First, draw line OC, OB and OA.

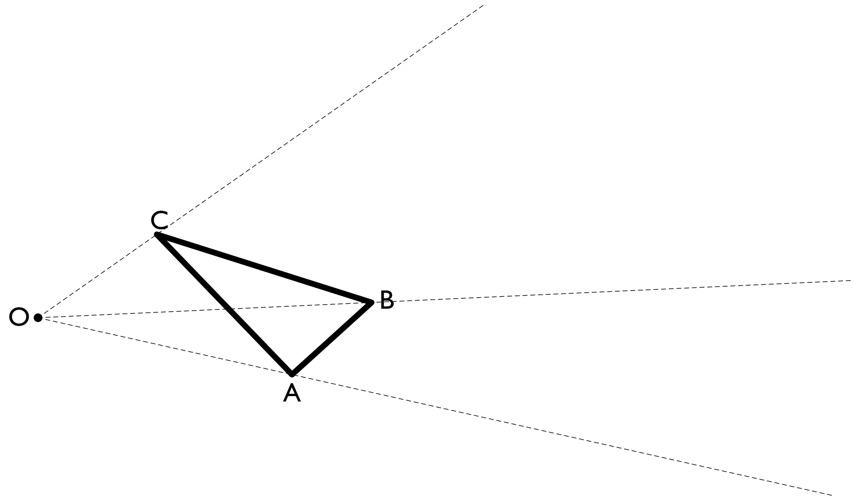


Figure 6.38. Dilation of a triangle by (O,3). STEP #1

Use compass, draw 3 arcs by OC, from the line OC. Do it with another 2 lines OB and OA. Then we get C', B' and A' as the shadow of C, B, and A. Give attention that |OC'| = 3 . |OC|, |OB'| = 3 . |OB|, and |OA'| = 3. |OA|. See figure below for more detail information.

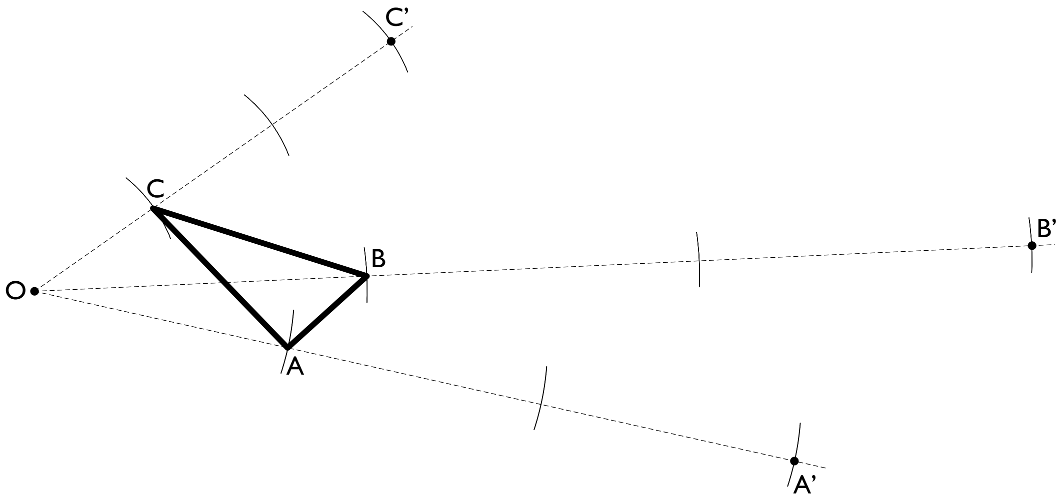


Figure 6.39. Dilation of a triangle by (O,3). STEP #2

The final step is just connecting A' to B', and also C'. We get the shadow of ∆ABC as shown below.

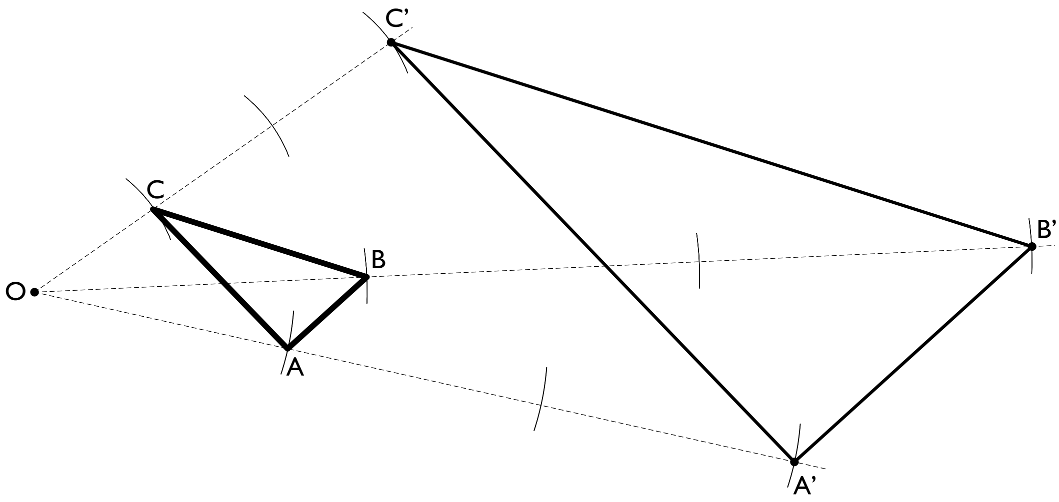


Figure 6.40. Dilation of a triangle by (O,3). The final result.

Now, we have explained the law of parallel lines.

D. SIMILARITY AS CONSEQUENCE OF PARALLEL LINES

We have deeply discussed about congruency on the chapter 3, but we may discuss again in this chapter some topics which has relation to parallel law. We know that 2 planar are congruent each other if there is a dilation.

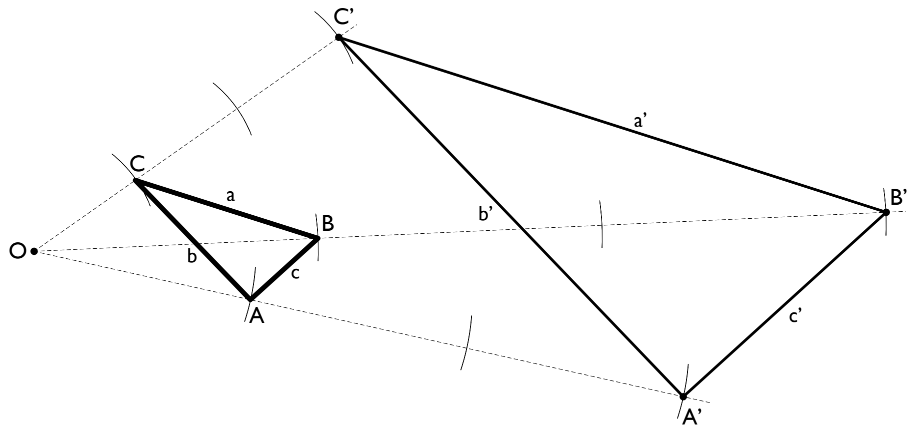


Figure 6.41. Similarity as product of Dilation.

In figure 6.41, we knew that ΔA'B'C' is dilate from ΔABC by dilate factor 3. The consequences from the theorems above are:

1. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$
2. m∠C = m∠C' ; m∠B = m∠B' ; m∠A = m∠A'
3. We call both of triangle are **similar**. We may write with:

ΔABC ≈ ΔA'B'C'

> **Definition 6.1.**
> ***The similarity of two triangles.***
> Let any ΔABC and ΔDEF. ΔABC similar to ΔDEF, denoted by ΔABC ≈ ΔDEF, if ΔABC is the shadow of ΔDEF or ΔABC dilate of ΔDEF by any dilate factor *r*.

Theorem!

6.11. Two triangles are called similar if all three corresponding side between both triangles has **comparison** each other. In this case, we call both triangles similar because of Side-side-side (SSS).

6.12. Two triangles are called similar if two parallel angles between both triangles have same size. In this case, we call both triangles similar because of Angle-Angle (AA).

6.13. Two triangles are called similar if two corresponding sides between both triangles have **comparison** each other and the adjacent angle has same size. In this case, we call both triangles similar because of Side-angle-side (SAS).

6.14. Two triangles are called similar, if both of triangles are right-angle triangle while the hypotenuse and one legs of triangle have **comparison** each other. In this case, we call both triangles similar because of Side-Hypotenuse (S-Hy).

See figure below for more detail explanation.

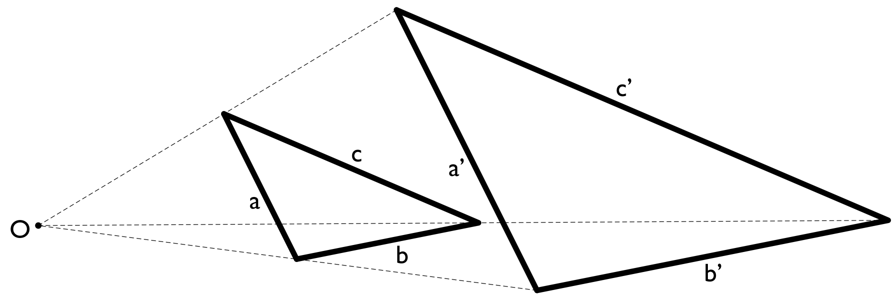


Figure 6.42. Condition of Th. 5.11. It is known $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = r$

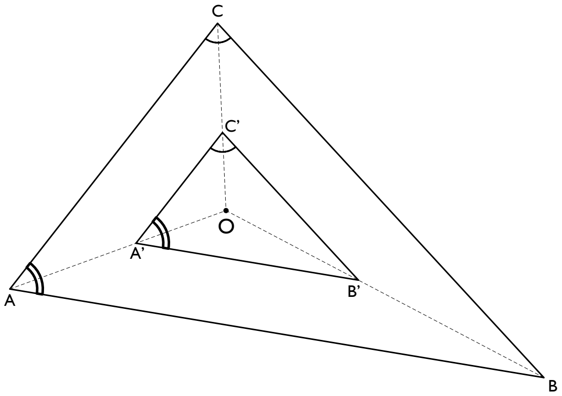


Figure 6.43. Condition of Th. 5.12. It is known m∠A = m∠A' ; m∠C = m∠C'

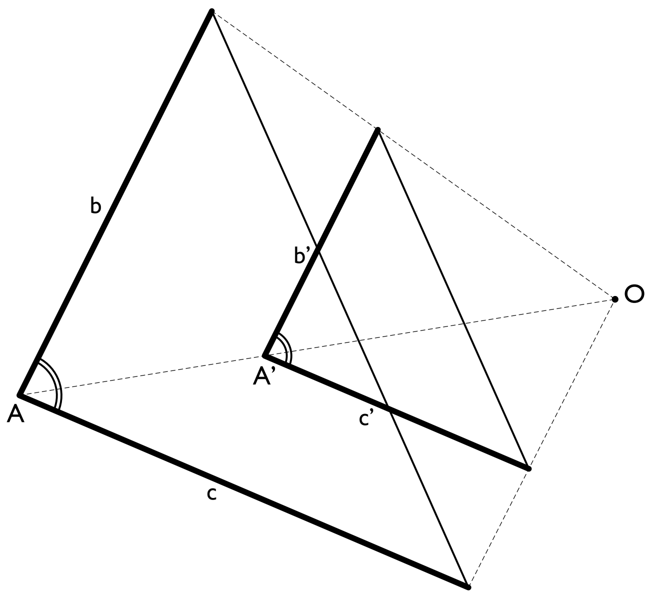


Figure 6.44. Condition of Th. 6.13. It is known m∠A = m∠A' and $\frac{b}{b'} = \frac{c}{c'}$

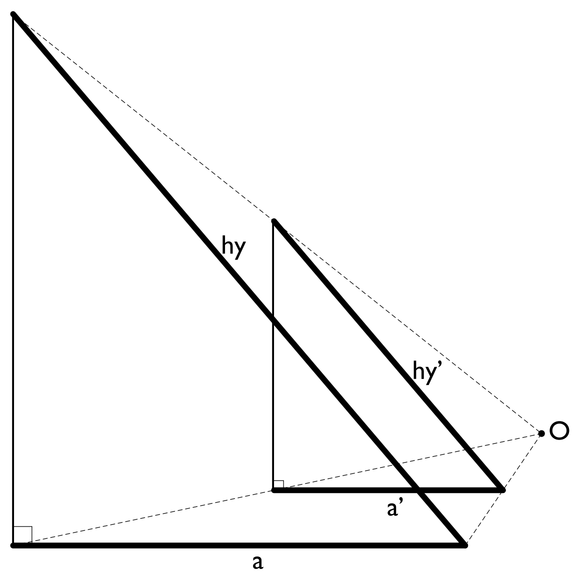


Figure 6.45. Condition of th. 5.14. It is known $\frac{a}{a\prime} = \frac{hy}{hy\prime}$

Proof!
We leave proof of theorem 6.11, 6.12, 6.13, and 6.14 as exercise for the reader.

We give an example as deeply understanding.

Example!
If AD and BE are altitudes of ΔABC, prove that CD × CB = CE × CA. See figure below.

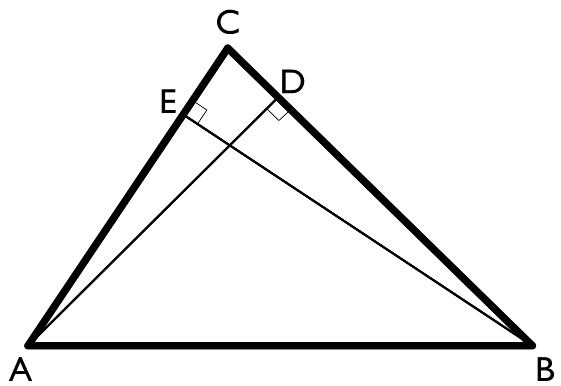


Figure 6.46. Condition of example problem

Solution!
Look at ΔACD and ΔBCE.
Obvious ∠ACD = ∠BCE, because of coincide, and
∠ADC = ∠BEC, because of perpendicular (90°).
We conclude that ΔACD ≈ ΔBCE.
The consequence is $\frac{CD}{CE} = \frac{CA}{CB}$ ⇔ CD × CB = CE × CA.
This solves the problem.

E. EXERCISE CHAPTER 6.

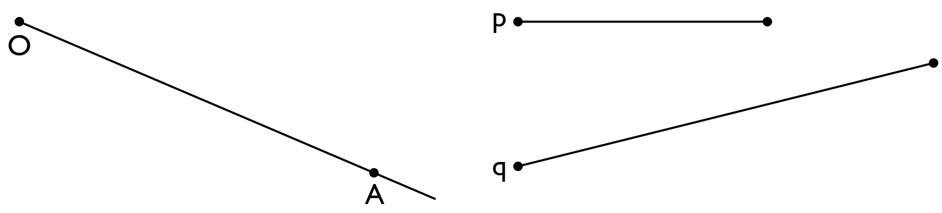


Figure 6.47. Lines for problem 1 and 2.

1. Draw vertex A' such that OA' = $\frac{5}{3}$. OA. See figure 6.47 above.
2. Dilate OA with $\left(0, -\frac{p}{q}\right)$, p and q are certain lines given.
3. On the ΔABC, AB = 16 cm, BC = 18 cm, AC = 12 cm. Vertex D is on AC such that AD = 8 cm. Draw vertex E on BC such that DE ∥ AB. After that draw vertex F on AB such that EF ∥ CA. Please mention the length of CE, CD, DE, and EF!
4. It is given ΔABC. On side BC there exist vertex D such that CD = $\frac{2}{5}$. BC and on the side AB there exist vertex E such that AE = $\frac{1}{3}$. AB. AD and CE cross each other and build cross point in S. Find the value of $\frac{AS}{SD}$ and $\frac{CS}{SE}$!
5. Divide a line be 5 similar parts!
6. Divide a line as parts such that each parts compare 2 : 5!
7. Divide a line such that each part has comparison as 2 lines p and q!
8. Find vertex C on the extension of line AB such that AB : BC = 7 : 3!
9. In the problems below, a, b, c, d, e, and f are certain lines. Draw:

(a) $x = \frac{ab}{c}$

(b) $x = \frac{b^2}{d}$

(c) $x = \frac{3ab}{2c}$

(d) $x = \frac{(c+d).c}{f}$

(e) $x = \frac{abc}{de}$

(f) $x = \frac{3b^2c}{ad}$

(g) $x = \frac{(a-b)^2}{c}$

Chapter 7

H I G H E R L E V E L T R I A N G L E

In this chapter we will learn some topics about higher level triangle. Those are projection theorem, Stewart theorem, and the next special line on triangle, Menealos and Ceva theorem. We assume that the reader have full knowledge about triangle which was learned before.

A. THEOREMS OF PROJECTION

We start explain about projection. Imagine that there are plane low flying at the mid day. We assume that the sun is exactly on the top of the plane. We say that the shadow of the plane is a projection of plane in earth.

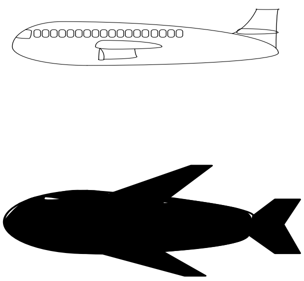


Figure 7.1. Plane and it's shadow

Let the plane as a ray, then we assumes that the sun light is perpendicular to earth, and the shadow is exactly under the plane, and then we can draw the condition on figure 7.1 as shown below.

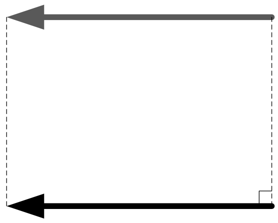


Figure 7.2. Plane and it's shadow

We assume that a **projection** has same direct with its **projectum** or object which is projected. We call a plane which is a projection held, by projection **screen**.

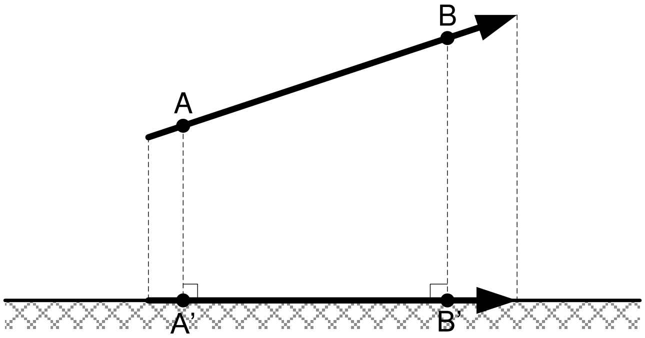


Figure 7.3. Projection process.

Figure 7.3 above show us the part of projection. We have known that ray AB is a projectum, ray A'B' is a projection on screen. We can call segment AA' and BB' be a **projector**. We often say that a projector is perpendicular to the screen. Now, we can write:

A'B' is projection of AB on the screen

We define properties of projection by some parts, those are:

1. Projection theorem on right-angle triangle

Base on the general properties of projection, we draw properties of projection on right-angle triangle as shown below.

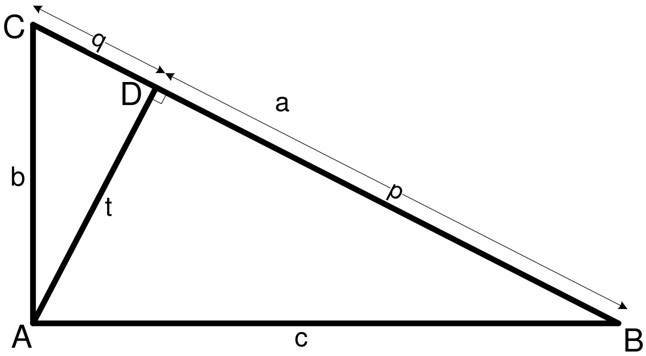


Figure 7.4. Projection on right-angle triangle.

Look at figure above! We can write some information about projection on that triangle, those are:

a. p is projection of c on side a,
b. q is projection of b on side a, and
c. t is the projector of its projection.

Theorems!

7.1. The square of one leg equals the multiplication product of its projection and hypotenuse of triangle. $b^2 = q.a$ *and* $c^2 = p.a$.

7.2. The square of the altitude to hypotenuse equals the multiplication product of its part. $t^2 = p.q$.

7.3. The product of legs multiplication equals to the multiplication of altitude and hypotenuse. $b.c = t.a$.

See figure below for more detail explanation.

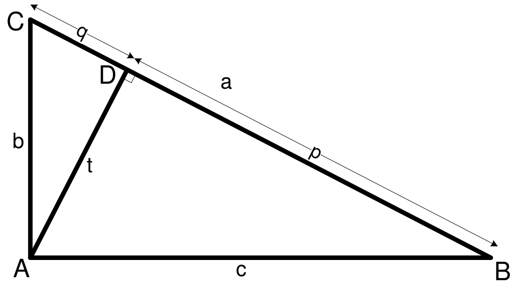


Figure 7.5. Projection condition on right-angle triangle.

Proof!
Look at ΔACD and ΔABC.
Obvious m∠ACD = m∠ACB, because of coincide, and
m∠ADC = m∠BAC, because of right angle,
From the fact above, we conclude that ΔACD ≈ ΔABC.
The direct consequence of the condition is $\frac{CD}{CA} = \frac{AD}{BA} = \frac{AC}{BC}$.
We get $\frac{CD}{CA} = \frac{AC}{BC} \Leftrightarrow \frac{q}{b} = \frac{b}{a} \Leftrightarrow b^2 = q.a$.
By similar step, we leave to proof $c^2 = p.a$ as the exercise to the reader. And then, these proves theorem 7.1. □

Look at ΔADC and ΔADB.
Because of ΔACD ≈ ΔABC, the consequence is also m∠CAD = m∠ABC. So that we can write some information below!
Obvious m∠CAD = m∠ABC, the consequence of ΔACD ≈ ΔABC,
m∠ADC = m∠BDA, because of right angle.
We conclude that ΔADC ≈ ΔADB.
The direct consequence of the condition is $\frac{AD}{BD} = \frac{CD}{AD} = \frac{AC}{AB}$.
We get $\frac{AD}{BD} = \frac{CD}{AD} \Leftrightarrow \frac{t}{p} = \frac{q}{t} \Leftrightarrow t^2 = p.q$.
These proves theorem 7.2. □

We proof theorem 7.3 by finding the area of triangle.
Obvious $A_{\Delta ABC} = \frac{1}{2}$. AB . BC = $\frac{1}{2}$. c . b and
$A_{\Delta ABC} = \frac{1}{2}$. AD . BC = $\frac{1}{2}$. t . a.
We get $\frac{1}{2}$. c . b = $\frac{1}{2}$. t . a ⇔ a.t = b.c.
This proves the theorems. □

2. Projection theorem on acute and obtuse triangle

In acute and obtuse triangles, it holds properties below.

Theorems!
7.4. We give two theorems below:
a. The square of length corresponding to **acute** angle equals the square sum of another two sides minus twice of multiplication its corresponding side and its projection.
b. The square of length corresponding to **obtuse** angle equals the square sum of another two sides plus twice of multiplication its corresponding side and its projection.

See figure below for more detail information!

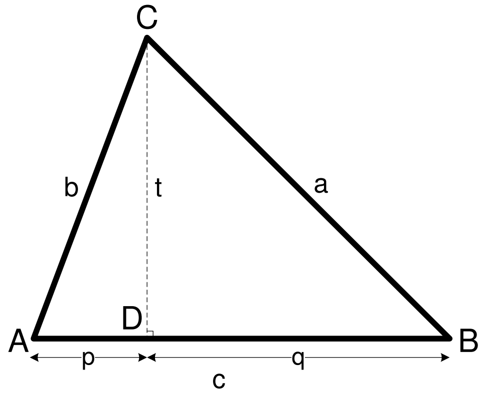


Figure 7.6. Projection condition on acute triangle.

Theorem 7.4.a. is just say that, in acute triangle there are obtaining two formulas below.

$$\mathbf{a^2 = b^2 + c^2 - 2.p.c} \qquad \text{......... 1*}$$

and

$$\mathbf{b^2 = a^2 + c^2 - 2.q.c} \qquad \text{......... 2*}$$

Proof!
At ΔADC, obvious $p^2 + t^2 = b^2 \Leftrightarrow t^2 = b^2 - p^2$ and
ΔDBC, obvious $q^2 + t^2 = a^2 \Leftrightarrow t^2 = a^2 - q^2$.
We get $b^2 - p^2 = a^2 - q^2$
$\Leftrightarrow a^2 = b^2 - p^2 + q^2$
$\Leftrightarrow a^2 = b^2 - p^2 + (c - p)^2$
$\Leftrightarrow a^2 = b^2 - p^2 + c^2 - 2pc + p^2$
$\Leftrightarrow a^2 = b^2 + c^2 - 2pc$.
Then we have proved 7.4.a. point 1* □

Obvious $b^2 - p^2 = a^2 - q^2$
$\Leftrightarrow b^2 = a^2 - q^2 + p^2$
$\Leftrightarrow b^2 = a^2 - q^2 + (c - q)^2$
$\Leftrightarrow b^2 = a^2 - q^2 + c^2 - 2qc + q^2$
$\Leftrightarrow b^2 = a^2 + c^2 - 2qc$
Then we have proved 7.4.a. point 2* □

Therefore, in obtuse triangle also hold properties below!

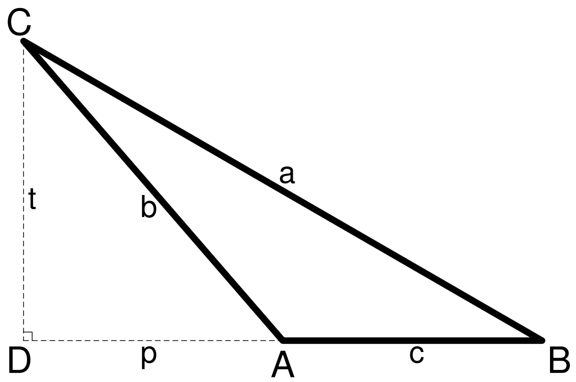


Figure 7.7. Projection condition on obtuse triangle.

Theorem 7.4.b. is just say that, in acute triangle there are obtaining two formulas below.

$$\mathbf{a^2 = b^2 + c^2 + 2.p.c} \qquad \text{………… } 3*$$

Proof!
Obvious $t^2 = b^2 - p^2$, because of Pythagoras theorem on $\triangle ACD$, and $t^2 = a^2 - (c + p)^2$, because of Pythagoras theorem on $\triangle BCD$.

Obvious $b^2 - p^2 = a^2 - (c + p)^2$
$\Leftrightarrow a^2 = b^2 - p^2 + (c + p)^2$
$\Leftrightarrow a^2 = b^2 - p^2 + c^2 + 2pc + p^2$
$\Leftrightarrow a^2 = b^2 + c^2 + 2pc.$
We get $a^2 = b^2 + c^2 + 2pc$.
This proves theorem 7.4.b. □

B. STEWART THEOREM

Theorem of Stewart!

7.5. If there is line x from C and dividing side c into c_1 and c_2 then hold formula below. (AD = c_1 and BD = c_2)

$$x^2c = a^2c_1 + b^2c_2 - c_1c_2c$$

See figure below for more detail information.

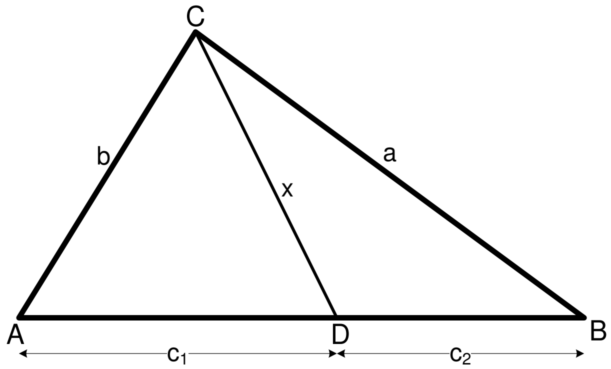


Figure 7.8. Condition on Stewart theorem.

Proof!
To proof the theorem, we need to build a helped line CE, such that CE ⊥ AB, and we label the length of segment DE with m. The figure as shown below!

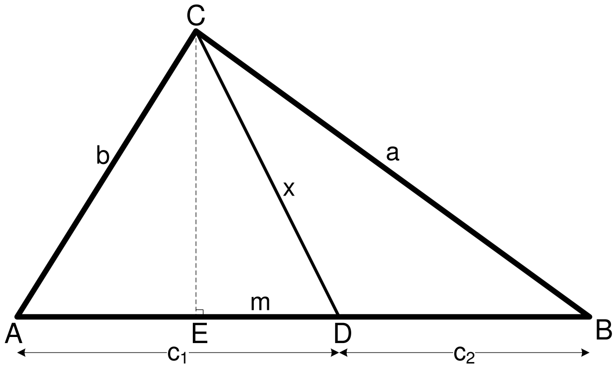


Figure 7.9. Proofing the Stewart theorem.

Look at ΔADC that is an acute triangle!
From theorem 7.4 we get

$b^2 = x^2 + c_1^2 - 2mc_1$

$\Leftrightarrow 2mc_1 = x^2 + c_1^2 - b^2$

$\Leftrightarrow m = \frac{x^2 + c_1{}^2 - b^2}{2c_1}$ …… 4∗

Look at ΔDBC, which is an obtuse triangle, angle BDA is obtuse angle.
From theorem 7.4 we get

$a^2 = x^2 + c_2^2 + 2mc_2$

$\Leftrightarrow -2mc_2 = x^2 + c_2^2 - a^2$

$\Leftrightarrow m = \frac{-x^2 - c_2^2 + a^2}{2c_2}$ …… 5∗

From 4∗ and 5∗ we get equation of *m* below.

$$\frac{x^2 + c_1{}^2 - b^2}{2c_1} = \frac{-x^2 - c_2^2 + a^2}{2c_2}$$

$$\Leftrightarrow c_2x^2 + c_2c_1{}^2 - c_2b^2 = -c_1x^2 - c_1c_2^2 + c_1a^2$$

$$\Leftrightarrow c_2x^2 + c_1x^2 = -c_1c_2^2 - c_2c_1{}^2 + c_2b^2 + c_1a^2$$

$$\Leftrightarrow (c_1 + c_2)x^2 = -c_1c_2(c_1 + c_2) + c_2b^2 + c_1a^2$$

$$\Leftrightarrow cx^2 = a^2c_1 + b^2c_2 - c_1c_2c$$

We have proved the formula is held for Stewart theorem. □

C. SPECIAL LINES ON TRIANGLE Part 2

We have discussed some special lines of a triangle. In this sub chapter we will a little deeper discuss the topics again. We discuss new theorem related to special lines of a triangle. Different from our topics in chapter 3, our discussion is not only about definition of each special lines of a triangle.

Here we remind the special lines of a triangle, those are a median, an altitude, and a bisector. Now, we give some properties about the special lines of a triangle.

Theorem!

7.6. The medians of a triangles is crossing each other and being divided into two parts by ratio 2 : 1

See figure below for more detail information.

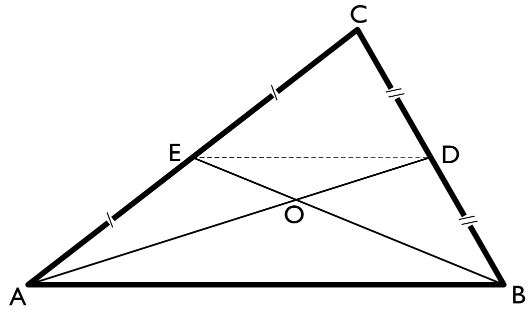


Figure 7.10. Condition of theorem 7.6.

It is given triangle ABC, AD and BE are two of its median. The theorem is just say that

$$\frac{\boldsymbol{AO}}{\boldsymbol{OD}} = \frac{\boldsymbol{BO}}{\boldsymbol{OE}} = \frac{\mathbf{2}}{\mathbf{1}}$$

Proof!

Because BE is median, then the consequence is CE : EA = 1 : 1, and because AD is median, then the consequence is CD : DB = 1 : 1.

Because CE : EA = CD : DB, from theorem 6.5[2] we conclude that DE ∥ AB.

Look at ΔCED and ΔCAB!

Obvious m∠DEC = m∠BAC, because of parallel angle, and
m∠DCE = m∠BCA, because of coincide.

So ΔCED ≈ ΔCAB.

The consequence of condition above is $\frac{CE}{CA} = \frac{CD}{CB} = \frac{DE}{AB} = \frac{2}{1}$.

Look at ΔDOE and ΔAOB!

Obvious m∠BAO = m∠EDO, because of interior alternate angle, and
m∠BOA = m∠DOE, because of opposite angle.

So ΔDOE ≈ ΔAOB.

The consequence of condition above is $\frac{BO}{OE} = \frac{AO}{OD} = \frac{DE}{AB} = \frac{2}{1}$

By similar step, you can prove if the median is build from vertex C.

These proved the theorem. □

[2] ***Theorem 6.5 :*** *If a line cross side AC and BC of a triangle in D and E, such that AD : DC = BE : EC , then the line DE is parallel to AB.*

Theorem!

7.7. If z_a, z_b, and z_c are median to side a, b, and c, then there is obtain formula below:

a. $z_a^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$

b. $z_b^2 = \frac{1}{2}a^2 + \frac{1}{2}c^2 - \frac{1}{4}b^2$

c. $z_c^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2$

Proof!

To proof the theorem, see figure below!

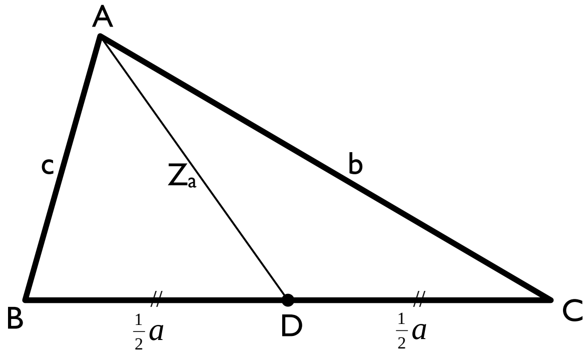


Figure 7.11. Condition of theorem 7.7.

From Stewart theorem, we get

$$z_a^2 a = b^2.\frac{1}{2}a + c^2.\frac{1}{2}a - \frac{1}{2}a.\frac{1}{2}a.a \Leftrightarrow z_a^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$$

By similar step, you can proof the formula obtain for z_b and z_c.

We leave the proof of z_b and z_c for exercise to the reader.

This proves the theorem. □

Theorem!

7.8. A bisector divides the corresponding side of angle bisected, by ratio like the adjacent sides.

See figure below for more detail information!

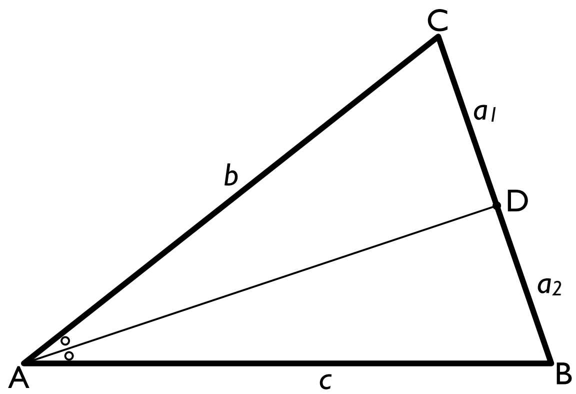


Figure 7.12. Condition of theorem 7.8.

It is given $\triangle ABC$. AD is an angle bisector of $\triangle ABC$.
The theorem said that

$$\frac{a_1}{a_2} = \frac{b}{c}$$

Proof!
To proof the theorem, we need helped lines as shown below.

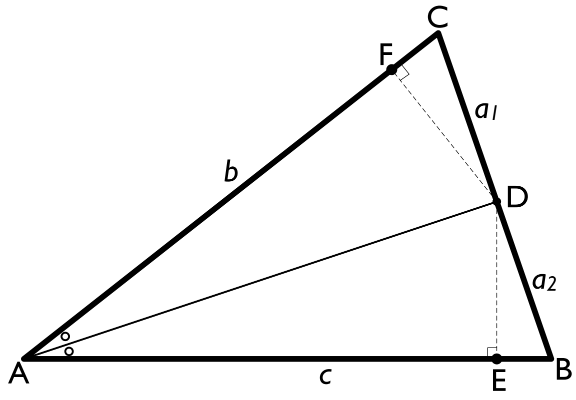


Figure 7.13. Proofing theorem 7.8.

Look at $\triangle ADF$ and $\triangle ADE$.
Obvious $m\angle EAD = m\angle FAD$, because of angle bisector AD,
$|AD| = |AD|$, because of coincidance, and
$m\angle ADF = m\angle ADE$, because the size of both angles are together $90° - \frac{1}{2}m\angle CAB$.
We conclude that $\triangle ADF \cong \triangle ADE$.
So that we get $|DE| = |DF|$.

Look at $\triangle ABD$ and $\triangle ACD$.
Obvious

$$\frac{A_{\Delta ABD}}{A_{\Delta ACD}} = \frac{\frac{1}{2}.AB.DE}{\frac{1}{2}.AC.DF} = \frac{AB}{AC}$$

and

$$\frac{A_{\Delta ABD}}{A_{\Delta ACD}} = \frac{\frac{1}{2}.a_2.AD}{\frac{1}{2}.a_1.AD} = \frac{a_2}{a_1}$$

We get $\frac{AB}{AC} = \frac{a_2}{a_1}$.
We write $AC : AB = a_1 : a_2$.
This proves the theorem. □

We assume theorem 7.8 also holds for outside angle bisector, like on figure below.

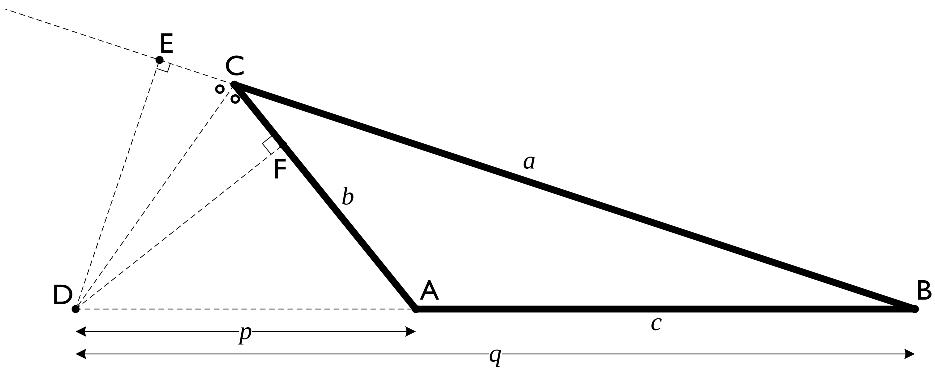


Figure 7.14. Condition of theorem 7.8 on outside angle bisector

Let p = |DA| and q = |DB|. DC is an outside angle bisector of ∠C. Figure 7.14 give us information that $p : q = b : a$.

Proof!
Obvious ΔFDC ≅ ΔEDC! (*why? Prove that!*) So we get |DE| = |DF|.
Look at ΔADC and ΔBDC.
Obvious $\frac{A_{\Delta ADC}}{A_{\Delta BDC}} = \frac{\frac{1}{2}.\mathrm{AC.DF}}{\frac{1}{2}.\mathrm{BC.DE}} = \frac{\mathrm{AC}}{\mathrm{BC}}$ and $\frac{A_{\Delta ADC}}{A_{\Delta BDC}} = \frac{\frac{1}{2}.p.t}{\frac{1}{2}.q.t} = \frac{p}{q}$
We get $\frac{\mathrm{AC}}{\mathrm{BC}} = \frac{p}{q}$.
We conclude that theorem holds for any condition of bisector. □

Theorem!
7.9. The square product of inside bisector equals the multiplication of two adjacent sides subtracted by parts of opposite side.

See figure below for more detail information!

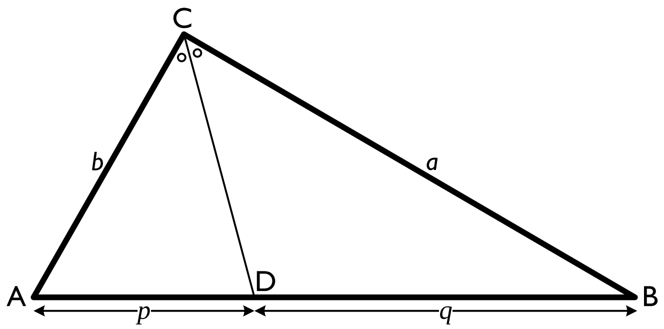


Figure 7.15. Condition of theorem 7.9 on inside angle bisector

Let ΔABC, CD is an angle bisector of C. We give label p for |DA| and q for |DB|. Theorem 7.9 is just say that $CD^2 = a.b - p.q$

Proof!
Because CD is a bisector then from theorem 7.8 we get $a : b = q : p$ or we write $ap = bq$.

From Stewart theorem[3] we get:

$CD^2.c = b^2.q + a^2p - pqc$

$\Leftrightarrow CD^2.c = b(bq) + a(ap) - pqc$

$\Leftrightarrow CD^2.c = b(ap) + a(bq) - pqc$

$\Leftrightarrow CD^2.c = abp + abq - pqc$

$\Leftrightarrow CD^2.c = ab\ (p + q) - pqc$

$\Leftrightarrow CD^2.c = ab\ c - pqc$

$\Leftrightarrow CD^2 = ab - pq$

We get $CD^2 = ab - pq$
And this proves the theorem. □

What about outside angle bisector? Does this formula obtain to that angle? See figure below! We will be shown that the formula is a little different with the formula above.

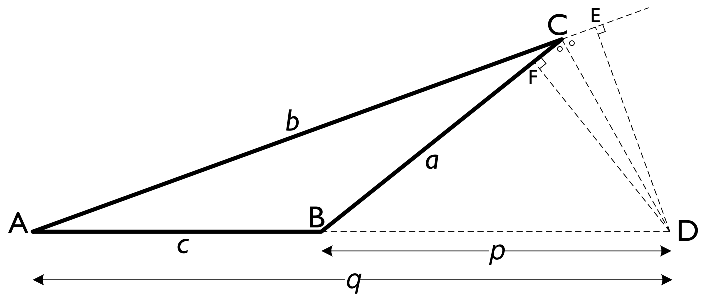


Figure 7.16. Condition of theorem 7.9 on outside angle bisector

Let ∆ABC, CD is outside bisector of ∠C. Let p = |DB| and q = |DA|.
We will prove that $CD^2 = pq - ab$.

Proof!
Because CD is a bisector then from theorem 7.8 we get $a : b = p : q$ or we write $aq = bp$.

From Stewart theorem, we get:

$a^2q = CD^2c + b^2p - pqc$

$\Leftrightarrow a(aq) = CD^2c + b(bp) - pqc$

$\Leftrightarrow a(bp) = CD^2c + b(aq) - pqc$

$\Leftrightarrow CD^2c = -abp + abq + pqc$

$\Leftrightarrow CD^2c = pqc - ab\ (p + q)$

$\Leftrightarrow CD^2c = pqc - ab\ c$

$\Leftrightarrow CD^2 = pq - ab$

We get $CD^2 = pq - ab$
We conclude that theorem holds for any condition of bisector. □

[3] *7.5 If there is line x from C and dividing side c into c_1 and c_2 then hold formula*
$\mathbf{x^2c = a^2c_1 + b^2c_2 - c_1c_2c}$

Theorem!

7.10. Let ΔABC. Let t_a and t_b are both altitudes that is build from side a and b. Then the comparison between both altitudes is reverse comparison of corresponding sides.

See figure below for more detail information!

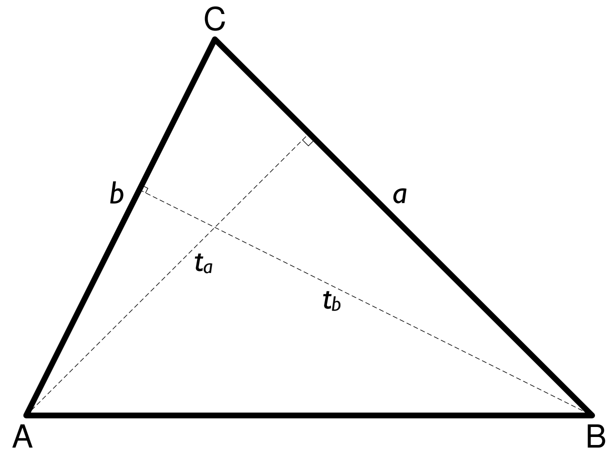


Figure 7.17. Condition of theorem 7.9 on outside angle bisector

It is given ΔABC. Let t_a and t_b are both altitudes that is build from side a and b.
The theorem is just said that

$$\frac{t_a}{t_b} = \frac{b}{a}$$

Proof!

Obvious $A_{\Delta ABC} = A_{\Delta ABC} \Leftrightarrow \frac{1}{2} \cdot b \cdot t_b = \frac{1}{2} \cdot a \cdot t_a \Leftrightarrow b \cdot t_b = a \cdot t_a$

We get $b \cdot t_b = a \cdot t_a \Leftrightarrow \frac{t_a}{t_b} = \frac{b}{a}$.

So we have proved the theorem. □

Theorem!

7.11. Let ΔABC. Let $2s = a + b + c$ and t_a , t_b , t_c are the altitudes drawn from side a, b, and c. We suggest the formula below hold for all triangles:

a. $t_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$

b. $t_b = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$

c. $t_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$

Proof!

We will show the formula is hold for t_a and we let the proof of t_b and t_c for exercise to the reader.

See figure below!

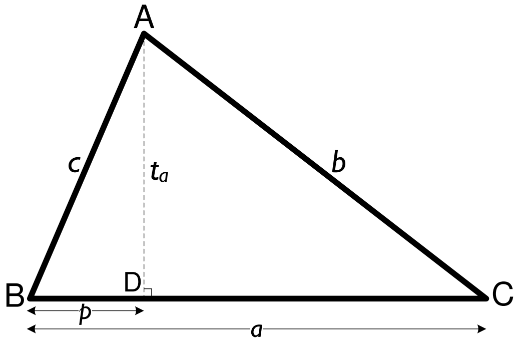


Figure 7.18. Condition of theorem 7.11 on t_a

Obvious $a + b + c = 2s \Leftrightarrow a + b - c = a + b + c - 2c = 2s - 2c = 2(s-c)$.
Obvious $a + b + c = 2s \Leftrightarrow b + c - a = a + b + c - 2a = 2s - 2a = 2(s-a)$.
Obvious $a + b + c = 2s \Leftrightarrow a + c - b = a + b + c - 2b = 2s - 2b = 2(s-b)$.

Obvious $t_a^2 = c^2 - p^2$, because of Pythagoras theorem on $\triangle ABD$.
Obvious $b^2 = c^2 + a^2 - 2ap \Leftrightarrow p = \frac{c^2+a^2-b^2}{2a}$, because of theorem 7.4.a.
We get

$t_a^2 = c^2 - p^2$

$\Leftrightarrow t_a^2 = c^2 - \left(\frac{c^2+a^2-b^2}{2a}\right)^2$

$\Leftrightarrow t_a^2 = \left(c - \frac{c^2+a^2-b^2}{2a}\right)\left(c + \frac{c^2+a^2-b^2}{2a}\right)$

$\Leftrightarrow t_a^2 = \left(\frac{2ac-c^2-a^2+b^2}{2a}\right)\left(\frac{2ac+c^2+a^2-b^2}{2a}\right)$

$\Leftrightarrow t_a^2 = \left(\frac{b^2-(a^2-2ac+c^2)}{2a}\right)\left(\frac{2ac+c^2+a^2-b^2}{2a}\right)$

$\Leftrightarrow t_a^2 = \left(\frac{b^2-(a-c)^2}{2a}\right)\left(\frac{(a+c)^2-b^2}{2a}\right)$

$\Leftrightarrow t_a^2 = \frac{(b-a+c)(b+a-c)(a+c+b)(a+c-b)}{4a^2}$

$\Leftrightarrow t_a^2 = \frac{(a+c+b)}{4a^2}\,(b + c - a)(a + b - c)(a + c - b)$

$\Leftrightarrow t_a^2 = \frac{2s}{4a^2}\,.\,2(s-a)\,.\,2(s-b)\,.\,2(s-c)$

$\Leftrightarrow t_a^2 = \frac{4}{a^2}\,.\,s(s-a)(s-b)(s-c)$

$\Leftrightarrow t_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$

So that we get $t_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$. Thus, the formula holds for t_b and t_c.
This proves the theorem. □

Look at figure below!

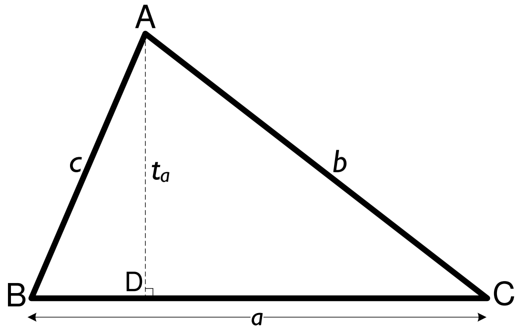


Figure 7.19. Consequence of theorem 7.11 on ΔABC

The direct consequence of the theorem 7.11 is

$$\mathrm{A}_{\Delta\mathrm{ABC}} = \frac{1}{2}.a.t_a = \frac{1}{2}.a.\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow \mathbf{A_{\Delta ABC}} = \sqrt{\mathbf{s(s-a)(s-b)(s-c)}}$$

And this is the end of sub topics of triangle. We have explained all properties of a triangle. Next sub chapter are exercises. We suggest to the reader to do all problem to increase the sense of your geometry.

Pythagoras,
is a mathematician which found a popular formula of a right-angle triangle $c^2 = a^2 + b^2$.

D. EXERCISE CHAPTER 7 #1

1. In the ΔABC, it is drawn the altitudes AD and BE. These lines intersects each other at H. Prove that $AH \times HD = BH \times HE$.
2. In the ΔABC, it is given AB = 10, BC = 12, and AC = 8. E is lying on BC such that CE = $4\frac{1}{2}$. D is lying on AC such that CD = 3. Prove that $\Delta CED \approx \Delta CAB$!
3. Two lines AB and CD are crossed at S, such that $SA \times SB = SC \times SD$. Draw lines BC and AD and prove that $m\angle ABC = m\angle CDA$, $m\angle BCD = m\angle BAD$, $m\angle CDB = m\angle BAC$, and $m\angle DBA = m\angle DCA$.
4. M is a midpoint of a hypotenuse BC of ΔABC. It is drawn two perpendicular lines MP and MQ, which P and Q are intersection points of perpendicular lines M on side AB and AC. Prove that $MA^2 = MP^2 \times MQ^2$.
5. In the ΔABC, it is known $m\angle A = 30°$, $m\angle B = 45°$, and $t_c = 8$. Find the length of its sides and also its area!
6. In the ΔABC, it is known $m\angle A = 90°$, $AD \perp BC$, $|AD| = 4$, $|BC| = 10$. Find the length of its right-legs!
7. In the ΔABC that is right-angle triangle, it is known $m\angle A = 90°$, prove that $\frac{1}{t_a{}^2} = \frac{1}{b^2} + \frac{1}{c^2}$!
8. In the ΔABC, it is known $m\angle A = 90°$, $AD \perp BC$, BD = 12, DC = 15. Find the length of AB, BC and also the area of ΔABC!
9. In the ΔABC, $|AB| = 13$, $|BC| = 20$, and $|AC| = 21$. Find the length of projection of BC to AB and projection to AC!
10. In the ΔABC, it is known $|AB| = 10$, $|BC| = 13$, $|AC| = 17$. At the extension of AB, there is lying a vertex P such that $|AP| : |PB| = 7 : 5$. Find the length of CP!
11. In the ΔABC, it is known $|AB| = 18$, $|BC| = 14$, $|AC| = 12$. D is lying on AB such that $|BD| = 8$. Find the length of CD!
12. In the ΔABC, it is known $|AB| = 40$, $|BC| = 42$, $|AC| = 26$. P is lying on AB and Q is lying on BC such that $|PB| = 16$ and $|QB| = 28$. Find the length of PQ!
13. In the isosceles triangle ABC (C is the vertex), there is any point D on its base and connected with C. Prove that $CD^2 = AC^2 - AD \times BD$
14. In the right-angle triangle ΔABC, C is right angle. On AB there lie point D such that $|AD| : |DB| = 1 : 2$. Prove that $9CD^2 = a^2 + 4b^2$!
15. The lengths of sides of ΔABC are 8, 10, and 12. Find the length of its medians!
16. Prove that in the ΔABC its hold formula :

$$4(z_a{}^2 + z_b{}^2 + z_c{}^2) = 3(a^2 + b^2 + c^2)$$

17. In the ΔABC, the altitudes AD and BE crosses each other at T. Find the length of AT and TD if it is known $a = 9$, $b = 7$, and $c = 8$!

18. The altitudes AD and BE of ΔABC crosses each other at T. Prove that $(AD \times AT) + (BT \times BE) = AB^2$!

19. In the ΔABC, it is build bisectors AD and BE. The lines cross at I. If $|AB| = 21$, $|BC| = 15$, and $|AC| = 24$, find the length of DI and IE!

20. In the ΔABC, $|AB| = 15$, $|BC| = 18$, and $|AC| = 12$. It is build bisector CD. Find the length of segment DE, E is midpoint of BC!

21. In the ΔABC, $|AB| = 12$, $|BC| = 6$, and $|AC| = 9$. Find the length of its altitudes, its medians, and inside also outside bisector of angle A!

22. In the ΔABC, |BC| is longer than |AC|. The inside and outside bisector of $\angle C$ is crossing AB (or the extension) in D and E. Prove that $BD \times AE = AD \times BE$!

E. MENELAUS THEOREM AND CEVA THEOREM

1. Two Rays

Before we continue our discussion to next topics, we will remind you about rays and its properties. We have introduced you about rays, but it's only about definition. We define ray AB is segment AB that is the direction if and only if in series A to B. We write ray AB by $\overrightarrow{AB}$ and it's seem like figure below.

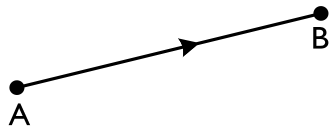


Figure 7.20. Ray AB or R_{AB} or $\overrightarrow{AB}$

Now, let there is $\overrightarrow{AB}$. We define $\overrightarrow{BA} = -\overrightarrow{AB}$. Obvious |BA| = |AB| but the direction between both of line is opposite each other. Thus, we can say the direction of $\overrightarrow{BA}$ is reverse of $\overrightarrow{AB}$. See figure below for more detail information.

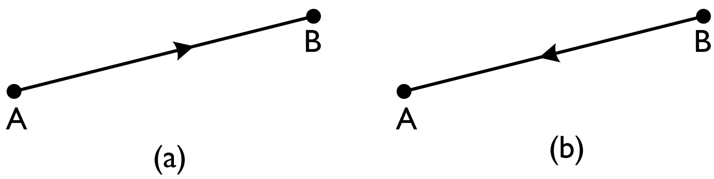


Figure 7.21. $\overrightarrow{AB}$ and $\overrightarrow{BA}$

Figure 7.21.a. show us $\overrightarrow{AB}$ and figure 7.21.b. show us $\overrightarrow{BA}$. Let there is a line and there are vertices A, B, C, and D lying on the line as shown below.

Figure 7.22.

Then, we can say that AB, AC, AD, BC, and CD have similar direction. Thus, DC, CB, DA, CA, and BA have opposite direction to the last. We assume direction to X_- has negative value and direction to X_+ has positive value. We give some note that |AB| = 3 = |BA|, but we are differencing the number $i.\overrightarrow{AB} = 3$ and $i.\overrightarrow{BA} = -3$. Another example is $i.\overrightarrow{AD} = 6$ and $i.\overrightarrow{DA} = -6$. The number 3, –3, 6, and –6 is called by **intro number.** Thus, we define intro number of $i.\overrightarrow{AB}$ as the number, which its value is similar to the distance AB.

Because intro numbers are also number, then we can operate the intro number with any operation that is hold for regular number. Here we give some properties of intro number operation.

Definition!
7.1. Relation of Chasles.

Let $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$ are rays.
Chasles relation of rays is define

$$i.\overrightarrow{AB} + i.\overrightarrow{BC} = i.\overrightarrow{AC}$$

To get more information to definition 7.1, please see figure below!

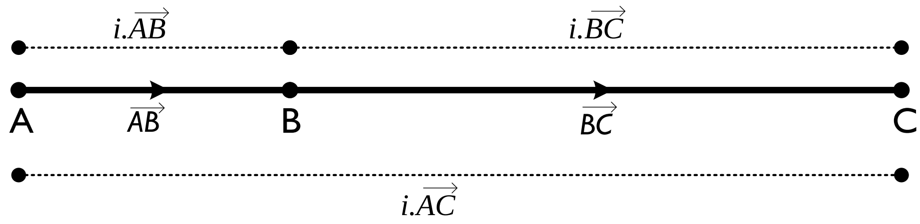


Figure 7.23. Condition of Chasles Relation

Figure 7.23 above show us $i.\overrightarrow{AC} = i.\overrightarrow{AB} + i.\overrightarrow{BC}$. That is |AC| = |AB| + |BC|, and then we give next properties.

Definition!
7.2. Relation of Mobius.

Let $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$ are rays.
Mobius relation of rays is define

$$i.\overrightarrow{AB} + i.\overrightarrow{BC} + i.\overrightarrow{CA} = i.\overrightarrow{AA}$$

$$\Leftrightarrow i.\overrightarrow{AB} + i.\overrightarrow{BC} + i.\overrightarrow{CA} = 0$$

Mobius relation is far-ranging form of Chasles relation. Mobius is introducing relation of rays, which the start point is equal to the finish point. The intro number for relation above is zero. See figure 7.22 for more detail information.

Figure 7.22 show us that is hold relation

$$i.\overrightarrow{AB} + i.\overrightarrow{BC} + i.\overrightarrow{CD} + i.\overrightarrow{DA} = 3 + 1 + 2 - 6 = 0$$

Thus, we can far-ranging definition 7.2 by formula below:

$$i.\overrightarrow{X_1X_2} + i.\overrightarrow{X_2X_3} + \ldots + i.\overrightarrow{X_{n-1}X_n} + i.\overrightarrow{X_nX_1} = 0$$

Let P is lie on line AB. If P lie between point A and B such that P divide segment AB from inside, then we call P **inside divisor** of AB become PA and PB. Thus, if P doesn't lie between A and B such that P divide segment AB from outside, then we call P **outside divisor** of AB. The comparison between $i.\overrightarrow{PA}$ and $i.\overrightarrow{PB}$ is called by **division ratio** of AB caused by point P. We denoted it with

$$(ABP) = \frac{i.\overrightarrow{PA}}{i.\overrightarrow{PB}}$$

See figure below for more detail information.

Figure 7.24. (a) P as inside divisor of AB and (b) P as outside divisor of AB

Attending to figure 7.24.a. and definition of intro number, we assume that the value of $i.\overrightarrow{PA}$ and $i.\overrightarrow{PB}$ is opposite each other, and then we get (*ABP*) would be always negative. Different with 7.24.a., on 7.24.b we assume that the direction of PA and PB is always same, and then we may write the value of (*ABP*) would be always positive. We assume that :

a. the ratio (*ABP*) is negative if P lie between A and B, and
b. the ratio (*ABP*) is positive if P is not lie between A and B.

2. Transversal

Let there is a polygon. A line m is called **transversal** if and only if m crosses the polygon. In fact if there are no sides, which is parallel to the transversal then the transversal is cross all side of the polygon. If a triangle is crossed by a transversal, then the transversal is crossing two of it sides and the extension of another one. Let see figure below for more detail explanation.

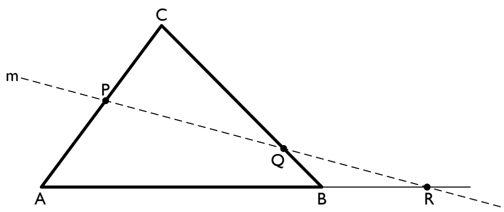


Figure 7.25. Line m is a transversal

The property of transversals is also hold even line m does not cross the triangle.

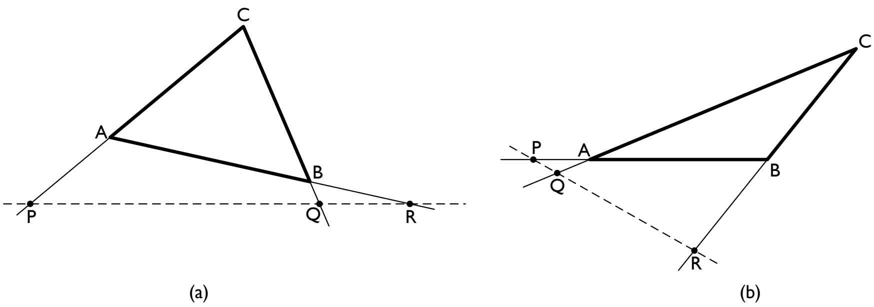


Figure 7.26. A transversal that is not through the triangle

Examples above are transversal hold for triangle. We assume that the properties hold for all polygons.

Can a transversal through a vertex of a polygon? The answer is yes. The transversal which is passing through a vertex or vertices of polygon is called **angle transversal**. Here we give an example use triangle.

Let there is ΔABC and three angle transversals, g, k, and m. If g, k, and m is intersecting each other at one point then the three angle transversals are crossing the opposite side of the divided angle. Thus, if there only one angle transversal crosses the side of triangle, then another two angle transversal should crosses the extension side of the triangle.

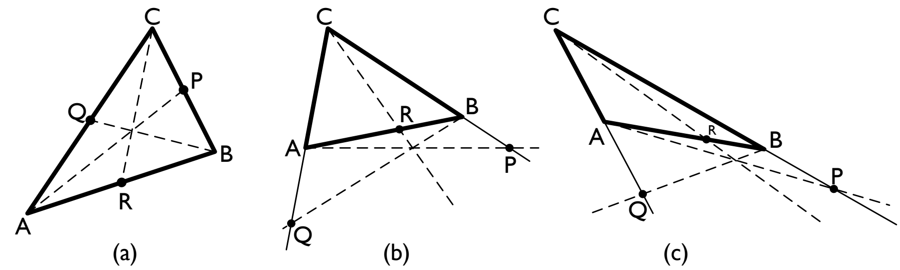


Figure 7.27. Angle transversals condition

3. Theorem of Menelaus

Theorem!

7.12. If a transversal of ∆ABC cross side AB, BC, and AC at point P, Q, and R then (*ABP*).(*BCQ*).(*CAR*) = 1.

See figure below for more detail explanation.

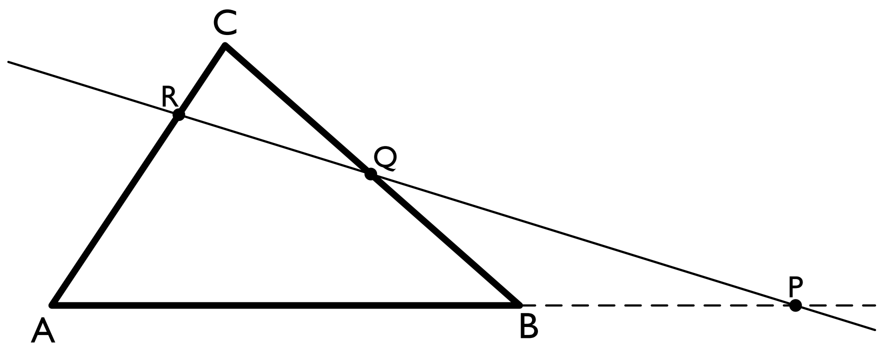


Figure 7.28. Menelaus theorem condition

Let there is ∆ABC, line PR is a transversal. The theorem said that

$$(ABP).(BCQ).(CAR) = 1$$

$$\Leftrightarrow \frac{i.\overrightarrow{PA}}{i.\overrightarrow{PB}}.\frac{i.\overrightarrow{QB}}{i.\overrightarrow{QC}}.\frac{i.\overrightarrow{RC}}{i.\overrightarrow{RA}} = 1$$

Proof!

Draw segment *a, b,* and *c*, such that *a* ∥ *b* ∥ *c*. Line *a* pass through vertex A, line *b* pass through vertex B, and also line *c*. See figure below!

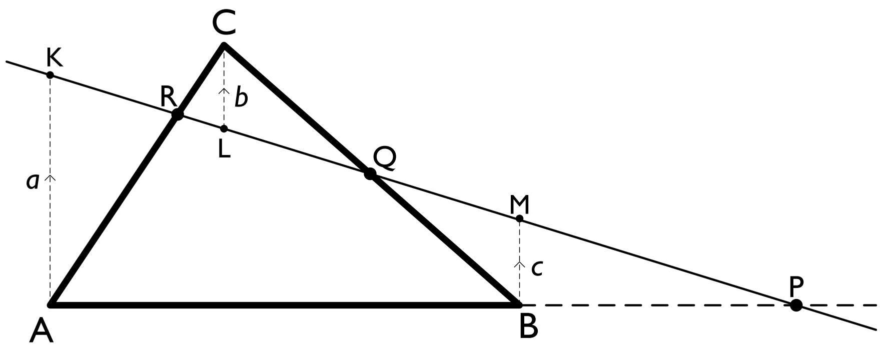


Figure 7.29. Proofing of Menelaus theorem

Look at ΔPAK and ΔPBM!
Obvious $\Delta PAK \approx \Delta PBM$ (*Why?*). So that we get $\frac{PA}{PB} = \frac{a}{c}$.
Because P doesn't lie between A and B, then we get

$$\frac{i.\overrightarrow{PA}}{i.\overrightarrow{PB}} = \frac{a}{c}$$

Look at ΔQBM and ΔQCL!
Obvious $\Delta QBM \approx \Delta QCL$ (*Why?*) So that we get $\frac{QB}{QC} = \frac{c}{b}$.
Because Q lies between B and C, then we get

$$\frac{i.\overrightarrow{QB}}{i.\overrightarrow{QC}} = -\frac{c}{b}$$

Look at ΔRAK and ΔRLC!
Obvious $\Delta RAK \approx \Delta RLC$ (*Why?*) So that we get $\frac{RC}{RA} = \frac{b}{a}$.
Because R lies between C and A, then we get

$$\frac{i.\overrightarrow{RC}}{i.\overrightarrow{RA}} = -\frac{b}{a}$$

Thus, from three equation above we get

$$\frac{i.\overrightarrow{PA}}{i.\overrightarrow{PB}} \times \frac{i.\overrightarrow{QB}}{i.\overrightarrow{QC}} \times \frac{i.\overrightarrow{RC}}{i.\overrightarrow{RA}} = \frac{a}{c} \times -\frac{c}{b} \times -\frac{b}{a} = 1$$

This proves Menelaus theorem. □

Thus, the reverse of Menelaus theorem also holds.

Theorem!

7.13. If on side AB, BC, and AC of a triangle ABC lies point P, Q, and R such that (*ABP*).(*BCQ*).(*CAR*) = 1 then the three points above lies on a similar line.

We leave the proof as exercise to the reader.
(Hints: use indirect proofing method to proof the theorem)

4. Theorem of Ceva

Ceva theorem shows us a property of transversal. Here we give a property about angle transversal.

Theorem!

7.14. Let there is ΔABC. Let g, k, and m are angle transversals which crosses the side AB, BC, and CA at P, Q, and R. If the three angle transversals is passing through one point then (*ABP*).(*BCQ*).(*CAR*) = −1.

See figure below for more detail explanation!

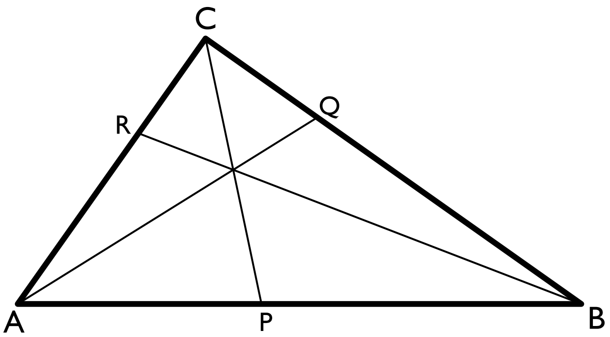


Figure 7.30. Ceva theorem condition

From figure above, Ceva was just want to explain that

$$(ABP).(BCQ).(CAR) = -1.$$

$$\Leftrightarrow \frac{i.\overrightarrow{PA}}{i.\overrightarrow{PB}}.\frac{i.\overrightarrow{QB}}{i.\overrightarrow{QC}}.\frac{i.\overrightarrow{RC}}{i.\overrightarrow{RA}} = -1$$

Proof!

Draw segment pass through vertex C, label it with z! Extends line AQ and BR until crosses the line z!

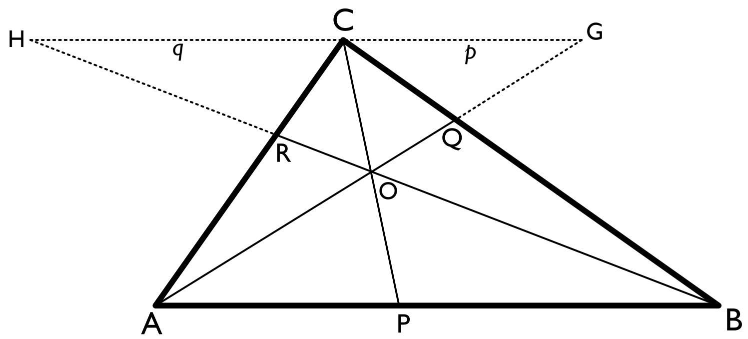


Figure 7.31. Proofing of Ceva theorem

Look at ΔAOB and ΔGOH!

Obvious Δ QGC ≅ ΔQAB (*Why?*). So that we get $\frac{PA}{PB} = \frac{p}{q}$.

Because P lies between A and B, then we get

$$\frac{\mathrm{i}.\overrightarrow{\mathrm{PA}}}{\mathrm{i}.\overrightarrow{\mathrm{PB}}} = -\frac{\mathrm{p}}{\mathrm{q}}$$

Look at ΔQGC and ΔQAB!

Obvious Δ QGC ≈ ΔQAB (*Why?*) So that we get $\frac{QB}{QC} = \frac{c}{p}$.

Because Q lies between B and C, then we get

$$\frac{\mathrm{i}.\overrightarrow{\mathrm{QB}}}{\mathrm{i}.\overrightarrow{\mathrm{QC}}} = -\frac{\mathrm{c}}{\mathrm{p}}$$

Look at ΔRCH and ΔRAB!

Obvious ∆RAK ≈ ∆RLC (*Why?*) So that we get $\frac{RC}{RA} = \frac{q}{c}$.
Because R lies between C and A, then we get

$$\frac{\mathrm{i.}\overrightarrow{RC}}{\mathrm{i.}\overrightarrow{RA}} = -\frac{q}{c}$$

Thus, from three equations above we get

$$\frac{\mathrm{i.}\overrightarrow{PA}}{\mathrm{i.}\overrightarrow{PB}} \times \frac{\mathrm{i.}\overrightarrow{QB}}{\mathrm{i.}\overrightarrow{QC}} \times \frac{\mathrm{i.}\overrightarrow{RC}}{\mathrm{i.}\overrightarrow{RA}} = -\frac{p}{q} \times -\frac{c}{p} \times -\frac{q}{c} = -1$$

This proves Ceva theorem. □

Thus, the reverse of Ceva theorem also holds.

Theorem!

7.15. If on side AB, BC, and AC of a triangle ABC lies point P, Q, and R such that (*ABP*).(*BCQ*).(*CAR*) = -1 then the three angle transversals of AQ, BR, and CP passes through one point.

We leave the proof as exercise to the reader.
(Hints: use indirect proofing method to proof the theorem)

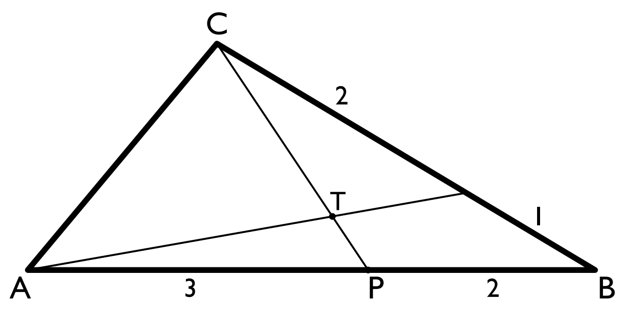
F. EXERCISE CHAPTER 7 #2

1. Prove that three medians of a triangle passes through one certain point and each median is divided to two segment with ratio 2 : 1 (starting from vertex)!

2. Prove that three bisectors pass through one certain point!

3. Prove that three altitudes pass through one certain point!

4. Find the comparison of CT : TP in the figure below!



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Ardhi Prabowo, born at February, 25th 1982, on small city Pekalongan. In 1991, he moved to Semarang and continued his education in Semarang. He finished the pre-graduate title in 2004 for mathematics education. After that he start his career to be a lecture on January 2005. Specialist at analysis and education program made him to be one of Olympics adviser on Central Java Province.

Graduate from Unnes on 2004 at mathematics education, and Magister from similar department on 2008, now he has 1 daughter as Alifiah Rizqeni. His wife was from similar department as Laely Rohmatin Apriliani.

Some research of him were about software 3D Studio Max which has relation to space geometry. Another research is about studying the application of software SWiSHmax into learning activity. The last research is about increasing spatial ability using software 3D studio max, was done in 2009.

His activity on English learning advising started in 2009, which was its content, was about advising the teacher teaching mathematics using English. On that activity he found that when teach mathematics the most important is make the student comfort. If student feel comfort, whatever the models, media and teaching aid, the material will be understood better.

As his residence, Sekargading Block O number 16 maybe the best place to take a rest, after daily activity. He also has residence in internet by ardhiprabowo@gmail.com. In his busy day, he also managed some blog like http://ardhiprabowo.blogspot.com or http://pakarbelajar.blogspot.com. He implemented the learning using blog and any other digital learning.