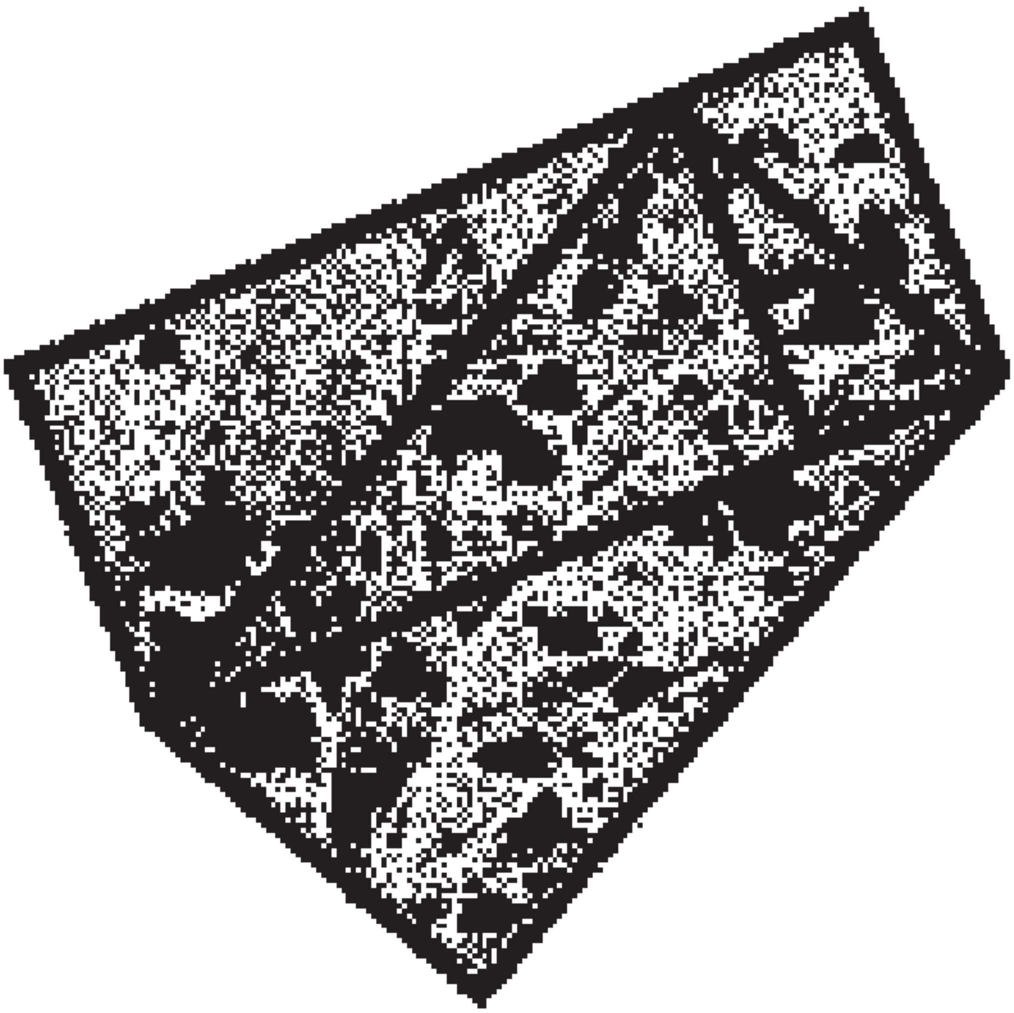
Kiselev's

GEOMETRY

Book I. PLANIMETRY



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by Alexander Givental

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Contents

Translator's Foreword

> *Those reading these lines are hereby summoned to raise their children to a good command of Elementary Geometry, to be judged by the rigorous standards of the ancient Greek mathematicians.*
>
> A magic spell

Mathematics is an ancient culture. It is passed on by each generation to the next. What we now call *Elementary Geometry* was created by Greeks some 2300 years ago and nurtured by them with pride for about a millennium. Then, for another millennium, Arabs were preserving Geometry and transcribing it to the language of *Algebra* that they invented. The effort bore fruit in the Modern Age, when exact sciences emerged through the work of Frenchman Rene Descartes, Englishman Isaac Newton, German Carl Friedrich Gauss, and their contemporaries and followers.

Here is one reason. On the decline of the 19th century, a Scottish professor showed to his class that the mathematical equations, he introduced to explain electricity experiments, admit wave-like solutions. Afterwards a German engineer Heinrich Hertz, who happened to be a student in that class, managed to generate and register the waves. A century later we find that almost every thing we use: GPS, TV, cell-phones, computers, and everything we manufacture, buy, or learn using them, descends from the mathematical discovery made by James Clerk Maxwell.

I gave the above speech at a graduation ceremony at the University of California Berkeley, addressing the class of graduating math majors — and then I cast a *spell* upon them.

Soon there came the realization that without a Magic Wand the spell won't work: I did not manage to find any textbook in English that I could recommend to a young person willing to master Elementary Geometry. This is when the thought of Kiselev's came to mind.

Andrei Petrovich Kiselev (pronounced And-'rei Pet-'ro-vich Ki-se-'lyov) left a unique legacy to mathematics education. Born in 1852 in a provincial Russian town Mzensk, he graduated in 1875 from the Department of Mathematics and Physics of St.-Petersburg University to begin a long career as a math and science teacher and author. His school-level textbooks "A Systematic Course of Arithmetic" [1] [9], "Elementary Algebra" [10], and "Elementary Geometry" (Book I "Planimetry", Book II "Stereometry") [3] were first published in 1884, 1888 and 1892 respectively, and soon gained a leading position in the Russian mathematics education. Revised and published more than a hundred times altogether, the books retained their leadership over many decades both in Tsarist Russia, and after the Revolution of 1917, under the quite different cultural circumstances of the Soviet epoch. A few years prior to Kiselev's death in 1940, his books were officially given the status of *stable*, i.e. main and only textbooks to be used in all schools to teach all teenagers in the totalitarian state with a 200-million population. The books held this status until 1955 (and "Stereometry" even until 1974) when they got replaced in this capacity by less successful clones written by more Soviet authors. Yet "Planimetry" remained the favorite under-the-desk choice of many teachers and a must for honors geometry students. In the last decade, Kiselev's "Geometry," which has long become a rarity, was reprinted by several major publishing houses in Moscow and St.-Petersburg in both versions: for teachers [6, 8] as an authentic pedagogical heritage, and for students [5, 7] as a textbook tailored to fit the currently active school curricula. In the post-Soviet educational market, Kiselev's "Geometry" continues to compete successfully with its own grandchildren.

What is the secret of such ageless vigor? There are several.

Kiselev himself formulated the following three key virtues of good textbooks: *precision*, *simplicity*, *conciseness*. And *competence in the subject* — for we must now add this fourth criterion, which could have been taken for granted a century ago.

Acquaintance with programs and principles of math education being developed by European mathematicians was another of Kiselev's assets. In his preface to the first edition of "Elementary Geometry," in addition to domestic and translated textbooks, Kiselev quotes ten geometry courses in French and German published in the previous decade.

Yet another vital elixir that prolongs the life of Kiselev's work was the continuous effort of the author himself and of the editors of later reprints to improve and update the books, and to accommodate the teachers' requests, curriculum fluctuations and pressures of the 20th century classroom.

Last but not least, deep and beautiful geometry is the most efficient preservative. Compared to the first textbook in this subject: the "Elements" [1], which was written by *Euclid of Alexandria* in the 3rd century B.C., and whose spirit and structure are so faithfully represented in Kiselev's "Geometry," the latter is quite young.

Elementary geometry occupies a singular place in secondary education. The acquiring of superb reasoning skills is one of those benefits from study-

[1] The numbers in brackets refer to the bibliography on p. 235.

ing geometry whose role reaches far beyond mathematics education *per se*. Another one is the unlimited opportunity for nurturing creative thinking (thanks to the astonishingly broad difficulty range of elementary geometry problems that have been accumulated over the decades). Fine learning habits of those who dared to face the challenge remain always at work for them. A lack thereof in those who missed it becomes hard to compensate by studying anything else. Above all, elementary geometry conveys the essence and power of the *theoretical method* in its purest, yet intuitively transparent and aesthetically appealing, form. Such high expectations seem to depend however on the appropriate framework: a textbook, a teacher, a culture.

In Russia, the adequate framework emerged apparently in the mid-thirties, with Kiselev's books as the key component. After the 2nd World War, countries of Eastern Europe and the Peoples Republic of China, adapted to their classrooms math textbooks based on Soviet programs. Thus, one way or another, Kiselev's "Geometry" has served several generations of students and teachers in a substantial portion of the planet. It is the time to make the book available to the English reader.

"Planimetry," targeting the age group of current 7–9th-graders, provides a concise yet crystal-clear presentation of elementary plane geometry, in all its aspects which usually appear in modern high-school geometry programs. The reader's mathematical maturity is gently advanced by commentaries on the nature of mathematical reasoning distributed wisely throughout the book. Student's competence is reinforced by generously supplied exercises of varying degree of challenge. Among them, *straightedge and compass* constructions play a prominent role, because, according to the author, they are essential for animating the subject and cultivating students' taste. The book is marked with the general sense of measure (in both selections and omissions), and non-cryptic, unambiguous language. This makes it equally suitable for independent study, teachers' professional development, or a regular school classroom. The book was indeed designed and tuned to be *stable*.

Hopefully the present adaptation retains the virtues of the original. I tried to follow it pretty closely, alternating between several available versions [3, 4, 5, 7, 8] when they disagreed. Yet authenticity of translation was not the goal, and I felt free to deviate from the source when the need occurred.

The most notable change is the significant extension and rearrangement of exercise sections to comply with the US tradition of making textbook editions self-contained (in Russia separate problem books are in fashion).

Also, I added or redesigned a few sections to represent material which found its way to geometry curricula rather recently.

Finally, having removed descriptions of several obsolete drafting devices (such as a pantograph), I would like to share with the reader the following observation.

In that remote, Kiselevian past, when Elementary Geometry was the most reliable ally of every engineer, the straightedge and compass were the main items in his or her drafting toolbox. The craft of blueprint drafting has long gone thanks to the advance of computers. Consequently, all 267 diagrams in the present edition are produced with the aid of graphing software *Xfig.* Still, Elementary Geometry is manifested in their design in multiple ways. Obviously, it is inherent in all modern technologies through the "custody chain": Euclid – Descartes – Newton – Maxwell. Plausibly, it awakened the innovative powers of the many scientists and engineers who invented and created computers. Possibly, it was among the skills of the authors of *Xfig.* Yet, symbolically enough, the most reliable way of drawing a diagram on the computer screen is to use electronic surrogates of the straightedge and compass and follow literally the prescriptions given in the present book, often in the very same theorem that the diagram illustrates. This brings us back to Euclid of Alexandria, who was the first to describe the theorem, and to the task of passing on *his* culture.

I believe that the book you are holding in your hands gives everyone a fair chance to share in the "custody." This is my Magic Wand, and now I can cast my spell.

Alexander Givental
Department of Mathematics
University of California Berkeley
April, 2006

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Hippocrates of Chios	*470 – 410 B.C.*
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Richard Dedekind	*1831 – 1916*
Ferdinand Lindemann	*1852 – 1939*
Heinrich Hertz	*1857 – 1894*

Introduction

1. Geometric figures. The part of space occupied by a physical object is called a **geometric solid**.

A geometric solid is separated from the surrounding space by a **surface**.

A part of the surface is separated from an adjacent part by a **line**.

A part of the line is separated from an adjacent part by a **point**.

The geometric solid, surface, line and point do not exist separately. However by way of abstraction we can consider a surface independently of the geometric solid, a line — independently of the surface, and the point — independently of the line. In doing so we should think of a surface as having no thickness, a line — as having neither thickness nor width, and a point — as having no length, no width, and no thickness.

A set of points, lines, surfaces, or solids positioned in a certain way in space is generally called a **geometric figure**. Geometric figures can move through space without change. Two geometric figures are called **congruent**, if by moving one of the figures it is possible to superimpose it onto the other so that the two figures become identified with each other in all their parts.

2. Geometry. A theory studying properties of geometric figures is called **geometry**, which translates from Greek as *land-measuring*. This name was given to the theory because the main purpose of geometry in antiquity was to measure distances and areas on the Earth's surface.

First concepts of geometry as well as their basic properties, are introduced as idealizations of the corresponding common notions and everyday experiences.

3. The plane. The most familiar of all surfaces is the flat surface, or the **plane**. The idea of the plane is conveyed by a window pane, or the water surface in a quiet pond.

We note the following property of the plane: *One can superimpose a plane on itself or any other plane in a way that takes one given point to any other given point, and this can also be done after flipping the plane upside down.*

4. The straight line. The most simple line is the **straight line**. The image of a thin thread stretched tight or a ray of light emitted through a small hole give an idea of what a straight line is. The following fundamental property of the straight line agrees well with these images:

For every two points in space, there is a straight line passing through them, and such a line is unique.

It follows from this property that:

If two straight lines are aligned with each other in such a way that two points of one line coincide with two points of the other, then the lines coincide in all their other points as well (because otherwise we would have two distinct straight lines passing through the same two points, which is impossible).

For the same reason, *two straight lines can intersect at most at one point.*

A straight line can lie in a plane. The following holds true:

If a straight line passes through two points of a plane, then all points of this line lie in this plane.

Figure 1 Figure 2 Figure 3

5. The unbounded straight line. Ray. Segment. Thinking of a straight line as extended indefinitely in both directions, one calls it an **infinite** (or **unbounded**) straight line.

A straight line is usually denoted by two uppercase letters marking any two points on it. One says "the line AB" or "BA" (Figure 1).

A part of the straight line bounded on both sides is called a **straight segment.** It is usually denoted by two letters marking its endpoints (the segment CD, Figure 2). Sometimes a straight line or a segment is denoted by one (lowercase) letter; one may say "the straight line a, the segment b."

Usually instead of "unbounded straight line" and "straight segment" we will simply say **line** and **segment** respectively.

Sometimes a straight line is considered which terminates in one direction only, for instance at the endpoint E (Figure 3). Such a straight line is called a **ray** (or **half-line**) drawn from E.

6. Congruent and non-congruent segments. *Two segments are congruent if they can be laid one onto the other so that their endpoints coincide.* Suppose for example that we put the segment AB onto the segment CD (Figure 4) by placing the point A at the point C and aligning the ray AB with the ray CD. If, as a result of this, the points B and D merge, then the segments AB and CD are congruent. Otherwise they are not congruent, and the one which makes a part of the other is considered smaller.

Figure 4

To mark on a line a segment congruent to a given segment, one uses the **compass**, a drafting device which we assume familiar to the reader.

7. Sum of segments. The sum of several given segments (AB, CD, EF, Figure 5) is a segment which is obtained as follows. On a line, pick any point M and starting from it mark a segment MN congruent to AB, then mark the segments NP congruent to CD, and PQ congruent to EF, both going in the same direction as MN. Then the segment MQ will be the sum of the segments AB, CD and EF (which are called **summands** of this sum). One can similarly obtain the sum of any number of segments.

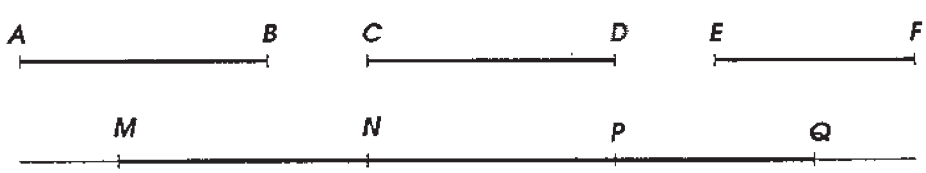

Figure 5

The sum of segments has the same properties as the sum of numbers. In particular it does not depend on the order of the summands (the **commutativity** law) and remains unchanged when some of the summands are replaced with their sum (the **associativity** law). For instance:

$$AB + CD + EF = AB + EF + CD = EF + CD + AB = \ldots$$

and

$$AB + CD + EF = AB + (CD + EF) = CD + (AB + EF) = \ldots.$$

8. Operations with segments. The concept of addition of segments gives rise to the concept of subtraction of segments, and multiplication and division of segments by a whole number. For example, the difference of AB and CD (if $AB > CD$) is a segment whose sum with CD is congruent to AB; the product of the segment AB with the number 3 is the sum of three segments each congruent to AB; the quotient of the segment AB by the number 3 is a third part of AB.

If given segments are measured by certain linear units (for instance, centimeters), and their lengths are expressed by the corresponding numbers, then the length of the sum of the segments is expressed by the sum of the numbers measuring these segments, the length of the difference is expressed by the difference of the numbers, etc.

9. The circle. If, setting the compass to an arbitrary step and, placing its pin leg at some point O of the plane (Figure 6), we begin to turn the compass around this point, then the other leg equipped with a pencil touching the plane will describe on the plane a continuous curved line all of whose points are the same distance away from O. This curved line is called a **circle**, and the point O — its **center**. A segment (OA, OB, OC in Figure 6) connecting the center with a point of the circle is called a **radius**. All radii of the same circle are congruent to each other.

Circles described by the compass set to the same radius are congruent because by placing their centers at the same point one will identify such circles with each other at all their points.

A line (MN, Figure 6) intersecting the circle at any two points is called a **secant**.

A segment (EF) both of whose endpoints lie on the circle is called a **chord**.

A chord (AD) passing through the center is called a **diameter**. A diameter is the sum of two radii, and therefore all diameters of the same circle are congruent to each other.

A part of a circle contained between any two points (for example, EmF) is called an **arc**.

The chord connecting the endpoints of an arc is said to **subtend** this arc.

An arc is sometimes denoted by the sign $\frown$; for instance, one writes: $\overset{\frown}{EmF}$.

The part of the plane bounded by a circle is called a **disk.**[2]

The part of a disk contained between two radii (the shaded part COB in Figure 6) is called a **sector**, and the part of the disk cut off by a secant (the part EmF) is called a **disk segment**.

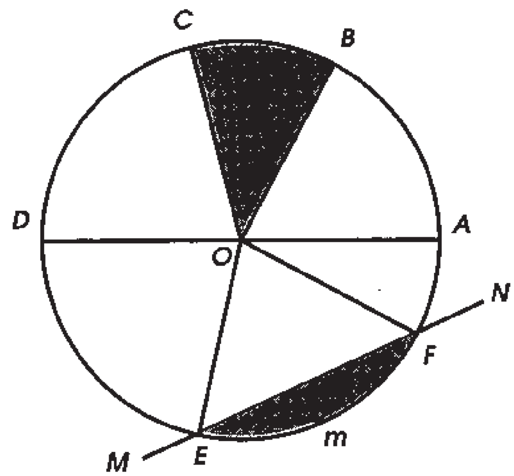

Figure 6

10. Congruent and non-congruent arcs. *Two arcs of the same circle (or of two congruent circles) are congruent if they can be aligned so that their endpoints coincide.* Indeed, suppose that we align the arc AB (Figure 7) with the arc CD by identifying the point A with the point C and directing the arc AB along the arc CD. If, as a result of this, the endpoints B and D coincide, then all the intermediate points of these arcs will coincide as well, since they are the same distance away from the center, and therefore $\overset{\frown}{AB}=\overset{\frown}{CD}$. But if B and D do not coincide, then the arcs are not congruent, and the one which is a part of the other is considered smaller.

11. Sum of arcs. The sum of several given arcs of the same radius is defined as an arc of that same radius which is composed from parts congruent respectively to the given arcs. Thus, pick an arbitrary point M (Figure 7) of the circle and mark the part MN

[2]Often the word "circle" is used instead of "disk." However one should avoid doing this since the use of the same term for different concepts may lead to mistakes.

congruent to AB. Next, moving in the same direction along the circle, mark the part NP congruent to CD. Then the arc MP will be the sum of the arcs AB and CD.

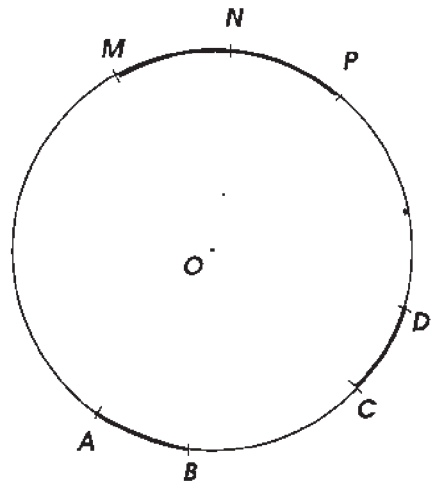

Figure 7

Adding arcs of the same radius one may encounter the situation when the sum of the arcs does not fit in the circle and one of the arcs partially covers another. In this case the sum will be an arc greater than the whole circle. For example, adding the arcs AmB and CnD (Figure 8) we obtain the arc consisting of the whole circle and the arc AD.

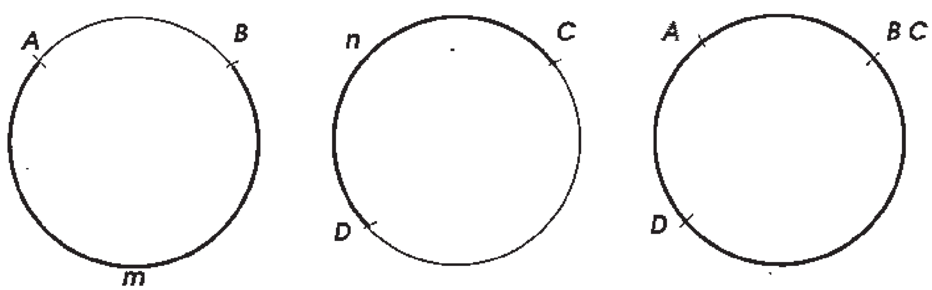

Figure 8

Similarly to addition of line segments, addition of arcs obeys the commutativity and associativity laws.

From the concept of addition of arcs one derives the concepts of subtraction of arcs, and multiplication and division of arcs by a whole number the same way as it was done for line segments.

12. Divisions of geometry. The subject of geometry can be divided into two parts: **plane geometry**, or **planimetry**, and **solid geometry**, or **stereometry**. Planimetry studies properties of those geometric figures all of whose elements fit the same plane.

EXERCISES

1. Give examples of geometric solids bounded by one, two, three, four planes (or parts of planes).

2. Show that if a geometric figure is congruent to another geometric figure, which is in its turn congruent to a third geometric figure, then the first geometric figure is congruent to the third.

3. Explain *why* two straight lines in space can intersect at most at one point.

4. Referring to §4, show that a plane not containing a given straight line can intersect it at most at one point.

*5.** [3] Give an example of a surface other than the plane which, like the plane, can be superimposed on itself in a way that takes any one given point to any other given point.
`Remark:` The required example is not unique.

6. Referring to §4, show that for any two points of a plane, there is a straight line lying *in this plane* and passing through them, and that such a line is unique.

7. Use a straightedge to draw a line passing through two points given on a sheet of paper. Figure out how to check that the line is really straight.
`Hint:` Flip the straightedge upside down.

*8.** Fold a sheet of paper and, using the previous problem, check that the edge is straight. Can you explain why the edge of a folded paper is straight?
`Remark:` There may exist several correct answers to this question.

9. Show that for each point lying in a plane there is a straight line lying in this plane and passing through this point. How many such lines are there?

10. Find surfaces other than the plane which, like the plane, together with each point lying on the surface contain a straight line passing through this point.
`Hint:` One can obtain such surfaces by bending a sheet of paper.

11. Referring to the definition of congruent figures given in §1, show that any two infinite straight lines are congruent; that any two rays are congruent.

12. On a given line, mark a segment congruent to four times a given segment, using a compass as few times as possible.

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[3] Stars * mark those exercises which we consider more difficult.

13. Is the sum (difference) of given segments unique? Give an example of two distinct segments which both are sums of the given segments. Show that these distinct segments are congruent.

14. Give an example of two non-congruent arcs whose endpoints coincide. Can such arcs belong to non-congruent circles? to congruent circles? to the same circle?

15. Give examples of non-congruent arcs subtended by congruent chords. Are there non-congruent chords subtending congruent arcs?

16. Describe explicitly the operations of subtraction of arcs, and multiplication and division of an arc by a whole number.

17. Follow the descriptions of operations with arcs, and show that multiplying a given arc by 3 and then dividing the result by 2, we obtain an arc congruent to the arc resulting from the same operations performed on the given arc in the reverse order.

18. Can sums (differences) of respectively congruent line segments, or arcs, be non-congruent? Can sums (differences) of respectively non-congruent segments, or arcs be congruent?

19. Following the definition of sum of segments or arcs, explain why addition of segments (or arcs) obeys the commutativity law.
Hint: Identify a segment (or arc) AB with BA.

Chapter 1

THE STRAIGHT LINE

1 Angles

13. Preliminary concepts. A figure formed by two rays drawn from the same point is called an **angle**. The rays which form the angle are called its **sides**, and their common endpoint is called the **vertex** of the angle. One should think of the sides as extending away from the vertex indefinitely.

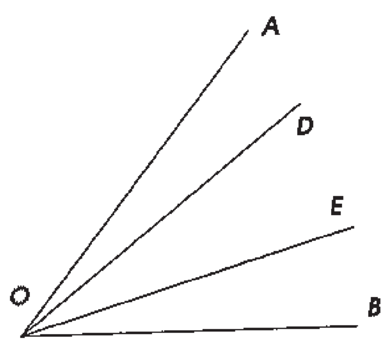

Figure 9

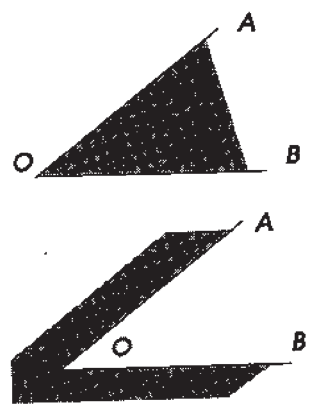

Figure 10

An angle is usually denoted by three uppercase letters of which the middle one marks the vertex, and the other two label a point on each of the sides. One says, e.g.: "the angle AOB" or "the angle BOA" (Figure 9). It is possible to denote an angle by one letter marking the vertex provided that no other angles with the same vertex are present on the diagram. Sometimes we will also denote an angle by a number placed inside the angle next to its vertex.

The sides of an angle divide the whole plane containing the angle into two regions. One of them is called the **interior** region of the angle, and the other is called the **exterior** one. Usually the interior region is considered the one that contains the segments joining any two points on the sides of the angle, e.g. the points A and B on the sides of the angle AOB (Figure 9). Sometimes however one needs to consider the other part of the plane as the interior one. In such cases a special comment will be made regarding which region of the plane is considered interior. Both cases are represented separately in Figure 10, where the interior region in each case is shaded.

Rays drawn from the vertex of an angle and lying in its interior (OD, OE, Figure 9) form new angles (AOD, DOE, EOB) which are considered to be parts of the angle (AOB).

In writing, the word "angle" is often replaced with the symbol $\angle$. For instance, instead of "angle AOB" one may write: $\angle AOB$.

14. Congruent and non-congruent angles. In accordance with the general definition of congruent figures (§1) *two angles are considered congruent if by moving one of them it is possible to identify it with the other.*

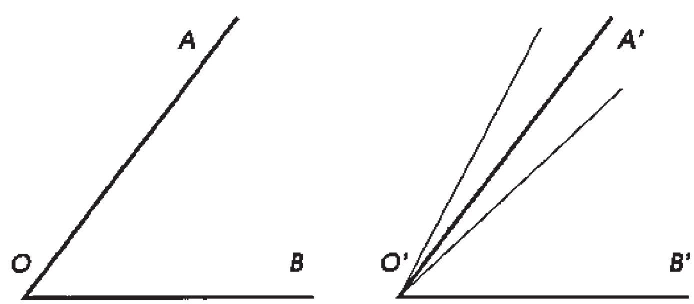

Figure 11

Suppose, for example, that we lay the angle AOB onto the angle $A'O'B'$ (Figure 11) in a way such that the vertex O coincides with O', the side OB goes along OB', and the interior regions of both angles lie on the same side of the line $O'B'$. If OA turns out to coincide with $O'A'$, then the angles are congruent. If OA turns out to lie inside or outside the angle $A'O'B'$, then the angles are non-congruent, and the one, that lies inside the other is said to be **smaller**.

15. Sum of angles. The sum of angles AOB and $A'O'B'$ (Figure 12) is an angle defined as follows. Construct an angle MNP congruent to the given angle AOB, and attach to it the angle PNQ, congruent to the given angle $A'O'B'$, as shown. Namely, the angle

MNP should have with the angle PNQ the same vertex N, a common side NP, and the interior regions of both angles should lie on the opposite sides of the common ray NP. Then the angle MNQ is called the sum of the angles AOB and $A'O'B'$. The interior region of the sum is considered the part of the plane comprised by the interior regions of the summands. This region contains the common side (NP) of the summands. One can similarly form the sum of three and more angles.

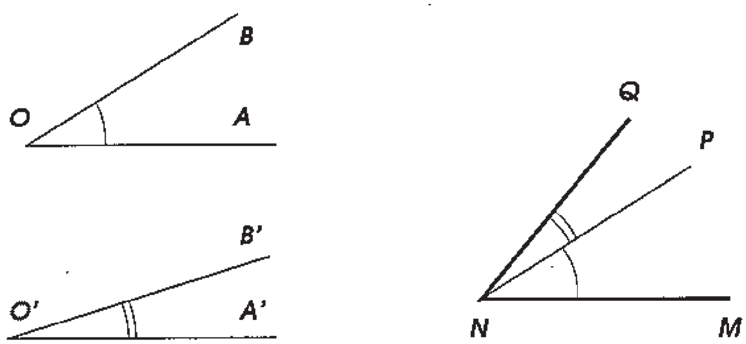

Figure 12

Addition of angles obeys the commutativity and associativity laws just the same way addition of segments does. From the concept of addition of angles one derives the concept of subtraction of angles, and multiplication and division of angles by a whole number.

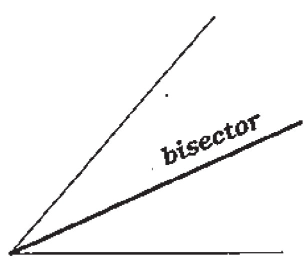

Figure 13

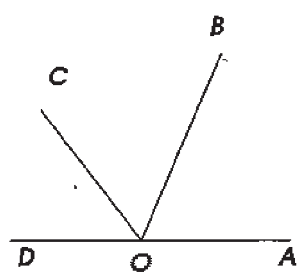

Figure 14

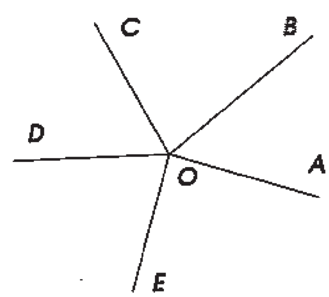

Figure 15

Very often one has to deal with the ray which divides a given angle into halves; this ray is called the **bisector** of the angle (Figure 13).

16. Extension of the concept of angle. When one computes the sum of angles some cases may occur which require special attention.

(1) It is possible that after addition of several angles, say, the

three angles: AOB, BOC and COD (Figure 14), the side OD of the angle COD will happen to be the continuation of the side OA of the angle AOB. We will obtain therefore the figure formed by two half-lines (OA and OD) drawn from the same point (O) and continuing each other. Such a figure is also considered an angle and is called a **straight angle**.

(2) It is possible that after the addition of several angles, say, the five angles: AOB, BOC, COD, DOE and EOA (Figure 15) the side OA of the angle EOA will happen to coincide with the side OA of the angle AOB. The figure formed by such rays (together with the whole plane surrounding the vertex O) is also considered an angle and is called a **full angle**.

(3) Finally, it is possible that added angles will not only fill in the whole plane around the common vertex, but will even overlap with each other, covering the plane around the common vertex for the second time, for the third time, and so on. Such an angle sum is congruent to one full angle added with another angle, or congruent to two full angles added with another angle, and so on.

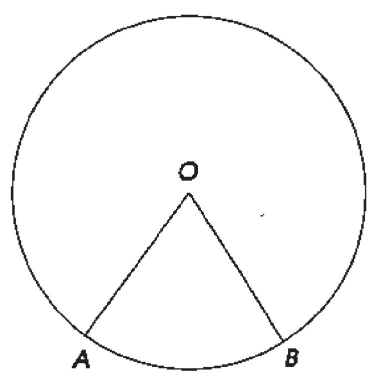

Figure 16

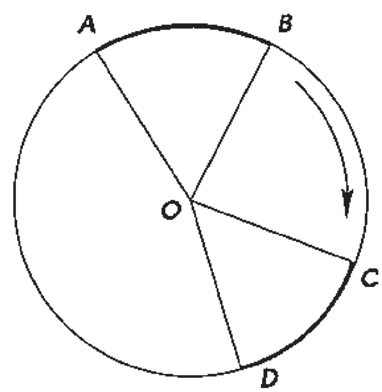

Figure 17

17. Central angle. The angle (AOB, Figure 16) formed by two radii of a circle is called a **central angle**; such an angle and the arc contained between the sides of this angle are said to *correspond* to each other.

Central angles and their corresponding arcs have the following properties.

In one circle, or two congruent circles:

(1) ***If central angles are congruent, then the corresponding arcs are congruent;***

(2) *Vice versa,* ***if the arcs are congruent, then the corre-***

sponding central angles are congruent.

Let $\angle AOB = \angle COD$ (Figure 17); we need to show that the arcs AB and CD are congruent too. Imagine that the sector AOB is rotated about the center O in the direction shown by the arrow until the radius OA coincides with OC. Then due to the congruence of the angles, the radius OB will coincide with OD; therefore the arcs AB and CD will coincide too, i.e. they are congruent.

The second property is established similarly.

18. Circular and angular degrees. Imagine that a circle is divided into 360 congruent parts and all the division points are connected with the center by radii. Then around the center, 360 central angles are formed which are congruent to each other as central angles corresponding to congruent arcs. Each of these arcs is called a **circular degree**, and each of those central angles is called an **angular degree**. Thus one can say that a circular degree is 1/360th part of the circle, and the angular degree is the central angle corresponding to it.

The degrees (both circular and angular) are further subdivided into 60 congruent parts called **minutes**, and the minutes are further subdivided into 60 congruent parts called **seconds**.

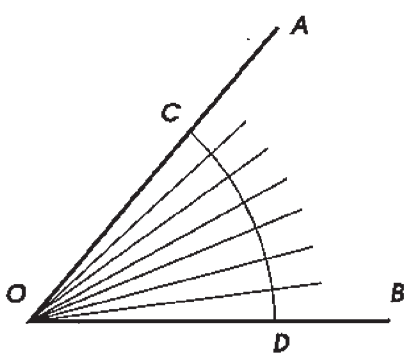

Figure 18

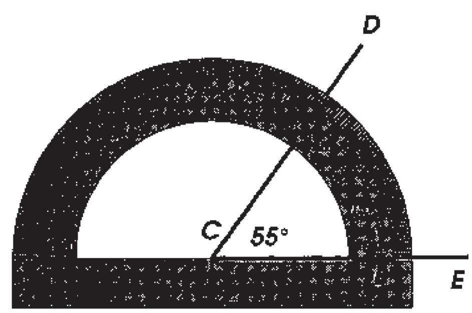

Figure 19

19. Correspondence between central angles and arcs. Let AOB be some angle (Figure 18). Between its sides, draw an arc CD of arbitrary radius with the center at the vertex O. Then the angle AOB will become the central angle corresponding to the arc CD. Suppose, for example, that this arc consists of 7 circular degrees (shown enlarged in Figure 18). Then the radii connecting the division points with the center obviously divide the angle AOB into 7 angular degrees. More generally, one can say that *an angle is measured by the arc corresponding to it*, meaning that an angle contains as many angular degrees, minutes and seconds as the corresponding arc contains circular degrees, minutes and seconds. For instance, if the arc CD contains 20 degrees 10 minutes and 15 seconds of circular units, then the angle AOB consists of 20 degrees 10 minutes and 15 seconds of angular units, which is customary to express as: $\angle AOB = 20°10'15''$, using the symbols °, ′ and ″ to denote degrees, minutes and seconds respectively.

Units of angular degree do not depend on the radius of the circle. Indeed, adding 360 angular degrees following the summation rule described in §15, we obtain the full angle at the center of the circle. Whatever the radius of the circle, this full angle will be the same. Thus one can say that an angular degree is 1/360th part of the full angle.

20. Protractor. This device (Figure 19) is used for measuring angles. It consists of a semi-disk whose arc is divided into 180°. To measure the angle DCE, one places the protractor onto the angle in a way such that the center of the semi-disk coincides with the vertex of the angle, and the radius CB lies on the side CE. Then the number of degrees in the arc contained between the sides of the angle DCE shows the measure of the angle. Using the protractor one can also draw an angle containing a given number of degrees (e.g. the angle of 90°, 45°, 30°, etc.).

EXERCISES

20. Draw any angle and, using a protractor and a straightedge, draw its bisector.

21. In the exterior of a given angle, draw another angle congruent to it. Can you do this in the interior of the given angle?

22. How many common sides can two distinct angles have?

23. Can two non-congruent angles contain 55 angular degrees each?

24. Can two non-congruent arcs contain 55 circular degrees each? What if these arcs have the same radius?

25. Two straight lines intersect at an angle containing 25°. Find the measures of the remaining three angles formed by these lines.

26. Three lines passing through the same point divide the plane into six angles. Two of them turned out to contain 25° and 55° respectively. Find the measures of the remaining four angles.

*27.** Using only compass, construct a 1° arc on a circle, if a 19° arc of this circle is given.

2 Perpendicular lines

21. Right, acute and obtuse angles. An angle of 90° (congruent therefore to one half of the straight angle or to one quarter of the full angle) is called a **right angle**. An angle smaller than the right one is called **acute**, and a greater than right but smaller than straight is called **obtuse** (Figure 20).

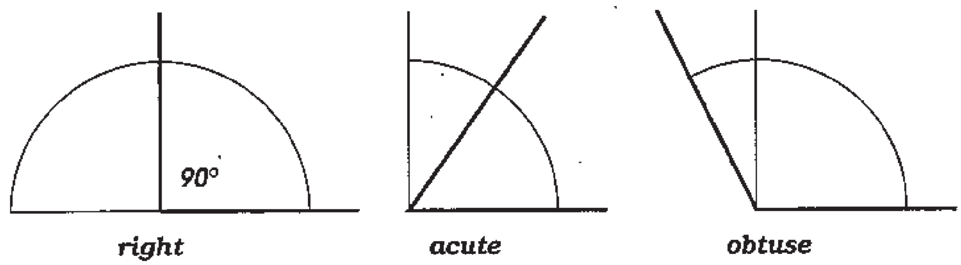

Figure 20

All right angles are, of course, *congruent to each other* since they contain the same number of degrees.

The measure of a right angle is sometimes denoted by d (the initial letter of the French word *droit* meaning "right").

22. Supplementary angles. Two angles (AOB and BOC, Figure 21) are called **supplementary** if they have one common side, and their remaining two sides form continuations of each other. Since the sum of such angles is a straight angle, *the sum of two supplementary angles is* 180° (in other words it is congruent to the sum of two right angles).

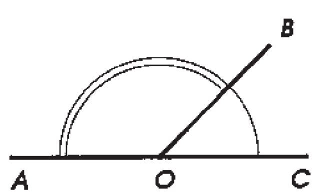

Figure 21

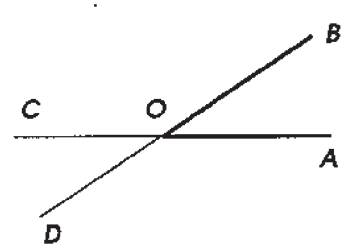

Figure 22

For each angle one can construct two supplementary angles. For example, for the angle AOB (Figure 22), prolonging the side AO we obtain one supplementary angle BOC, and prolonging the side BO we obtain another supplementary angle AOD. *Two angles supplementary to the same one are congruent to each other*, since they both contain the same number of degrees, namely the number that supplements the number of degrees in the angle AOB to 180° contained in a straight angle.

If AOB is a right angle (Figure 23), i.e. if it contains 90°, then each of its supplementary angles COB and AOD must also be right, since it contains 180° – 90°, i.e. 90°. The fourth angle COD has to be right as well, since the three angles AOB, BOC and AOD contain 270° altogether, and therefore what is left from 360° for the fourth angle COD is 90° too. Thus, *if one of the four angles formed by two intersecting lines* (AC and BD, Figure 23) *is right, then the other three angles must be right as well.*

23. A perpendicular and a slant. In the case when two supplementary angles are not congruent to each other, their common side (OB, Figure 24) is called a **slant** [1] to the line (AC) containing the other two sides. When, however, the supplementary angles are congruent (Figure 25) and when, therefore, each of the angles is right, the common side is called a **perpendicular** to the line containing the other two sides. The common vertex (O) is called the **foot of the slant** in the first case, and the **foot of the perpendicular** in the second.

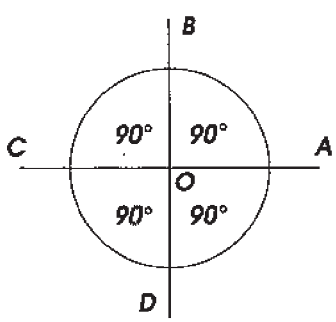

Figure 23

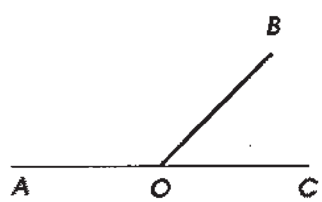

Figure 24

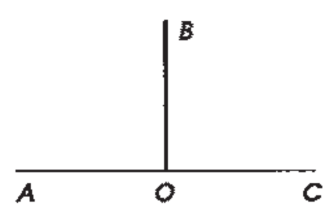

Figure 25

Two lines (AC and BD, Figure 23) intersecting at a right angle are called **perpendicular** to each other. The fact that the line AC is perpendicular to the line BD is written: $AC \perp BD$.

Remarks. (1) If a perpendicular to a line AC (Figure 25) needs to be drawn through a point O lying on this line, then the perpendicular is said to be "erected" to the line AC, and if the perpendicular needs to be drawn through a point B lying outside the line, then the perpendicular is said to be "dropped" to the line (no matter if it is upward, downward or sideways).

[1] Another name used for a slant is an **oblique line**.

(2) Obviously, at any given point of a given line, on either side of it, one can erect a perpendicular, and such a perpendicular is unique.

24. Let us prove that ***from any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.***

Let a line AB (Figure 26) and an arbitrary point M outside the line be given. We need to show that, first, one can drop a perpendicular from this point to AB, and second, that there is only one such perpendicular.

Imagine that the diagram is folded so that the upper part of it is identified with the lower part. Then the point M will take some position N. Mark this position, unfold the diagram to the initial form and then connect the points M and N by a line. Let us show now that the resulting line MN is perpendicular to AB, and that any other line passing through M, for example MD, is not perpendicular to AB. For this, fold the diagram again. Then the point M will merge with N again, and the points C and D will remain in their places. Therefore the line MC will be identified with NC, and MD with ND. It follows that $\angle MCB = \angle BCN$ and $\angle MDC = \angle CDN$.

But the angles MCB and BCN are supplementary. Therefore each of them is right, and hence $MN \perp AB$. Since MDN is not a straight line (because there can be no two straight lines connecting the points M and N), then the sum of the two congruent angles MDC and CDN is not equal to $2d$. Therefore the angle MDC is not right, and hence MD is not perpendicular to AB. Thus one can drop no other perpendicular from the point M to the line AB.

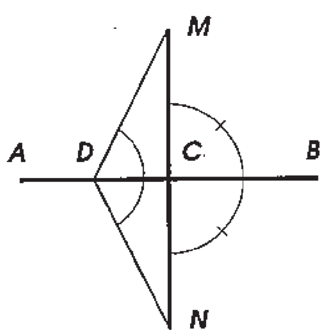

Figure 26

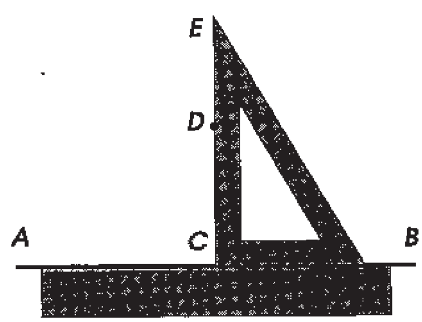

Figure 27

25. The drafting triangle. For practical construction of a perpendicular to a given line it is convenient to use a **drafting triangle** made to have one of its angles right. To draw the perpendicular to a line AB (Figure 27) through a point C lying on this line, or through a point D taken outside of this line, one can align a straightedge with the line AB, the drafting triangle with the straightedge, and then slide the triangle along the straightedge until the other side of the right angle hits the point C or D, and then draw the line CE.

26. Vertical angles. Two angles are called **vertical** if the sides of one of them form continuations of the sides of the other. For instance, at the intersection of two lines AB and CD (Figure 28) two pairs of vertical angles are formed: AOD and COB, AOC and DOB (and four pairs of supplementary angles).

Two vertical angles are congruent to each other (for example, $\angle AOD = \angle BOC$) since each of them is supplementary to the same angle (to $\angle DOB$ or to $\angle AOC$), and such angles, as we have seen (§22), are congruent to each other.

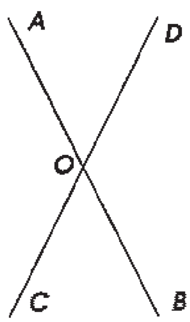

Figure 28

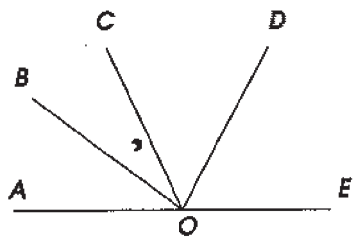

Figure 29

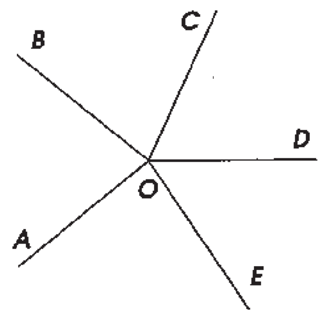

Figure 30

27. Angles that have a common vertex. It is useful to remember the following simple facts about angles that have a common vertex:

(1) *If the sum of several angles* (AOB, BOC, COD, DOE, Figure 29) *that have a common vertex is congruent to a straight angle, then the sum is* $2d$, *i.e.* $180°$.

(2) *If the sum of several angles* (AOB, BOC, COD, DOE, EOA, Figure 30) *that have a common vertex is congruent to the full angle, then it is* $4d$, *i.e.* $360°$.

(3) *If two angles* (AOB and BOC, Figure 24) *have a common vertex* (O) *and a common side* (OB) *and add up to* $2d$ (i.e. $180°$), *then their two other sides* (AO and OC) *form continuations of each other* (i.e. such angles are supplementary).

EXERCISES

28. Is the sum of the angles $14°24'44''$ and $75°35'25''$ acute or obtuse?

29. Five rays drawn from the same point divide the full angle into five congruent parts. How many different angles do these five rays form? Which of these angles are congruent to each other? Which of them are acute? Obtuse? Find the degree measure of each of them.

30. Can both angles, whose sum is the straight angle, be acute? obtuse?

31. Find the smallest number of acute (or obtuse) angles which add up to the full angle.

32. An angle measures $38°20'$; find the measure of its supplementary angles.

33. One of the angles formed by two intersecting lines is $2d/5$. Find the measures of the other three.

34. Find the measure of an angle which is congruent to twice its supplementary one.

35. Two angles ABC and CBD having the common vertex B and the common side BC are positioned in such a way that they do not cover one another. The angle $ABC = 100°20'$, and the angle $CBD = 79°40'$. Do the sides AB and BD form a straight line or a bent one?

36. Two distinct rays, perpendicular to a given line, are erected at a given point. Find the measure of the angle between these rays.

37. In the interior of an obtuse angle, two perpendiculars to its sides are erected at the vertex. Find the measure of the obtuse angle, if the angle between the perpendiculars is $4d/5$.

Prove:

38. Bisectors of two supplementary angles are perpendicular to each other.

39. Bisectors of two vertical angles are continuations of each other.

40. If at a point O of the line AB (Figure 28) two congruent angles AOD and BOC are built on the opposites sides of AB, then their sides OD and OC form a straight line.

41. If from the point O (Figure 28) rays OA, OB, OC and OD are constructed in such a way that $\angle AOC = \angle DOB$ and $\angle AOD = \angle COB$, then OB is the continuation of OA, and OD is the continuation of OC.
Hint: Apply §27, statements 2 and 3.

3 Mathematical propositions

28. Theorems, axioms, definitions. From what we have said so far one can conclude that some geometric statements we consider quite obvious (for example, the properties of planes and lines in §3 and §4) while some others are established by way of reasoning (for example, the properties of supplementary angles in §22 and vertical angles in §26). In geometry, this process of reasoning is a principal way to discover properties of geometric figures. It would be instructive therefore to acquaint yourself with the forms of reasoning usual in geometry.

All facts established in geometry are expressed in the form of propositions. These propositions are divided into the following types.

Definitions. Definitions are propositions which explain what meaning one attributes to a name or expression. For instance, we have already encountered the definitions of central angle, right angle, perpendicular lines, etc.

Axioms. Axioms [2] are those facts which are accepted without proof. This includes, for example, some propositions we encountered earlier (§4): through any two points there is a unique line; if two points of a line lie in a given plane then all points of this line lie in the same plane.

Let us also mention the following axioms which apply to any kind of quantities:

if each of two quantities is equal to a third quantity, then these two quantities are equal to each other;

if the same quantity is added to or subtracted from equal quantities, then the equality remains true;

if the same quantity is added to or subtracted from unequal quantities, then the inequality remains unchanged, i.e. the greater quantity remains greater.

Theorems. Theorems are those propositions whose truth is found only through a certain reasoning process (proof). The following propositions may serve as examples:

if in one circle or two congruent circles some central angles are congruent, then the corresponding arcs are congruent;

if one of the four angles formed by two intersecting lines turns out to be right, then the remaining three angles are right as well.

[2] In geometry, some axioms are traditionally called **postulates**.

Corollaries. Corollaries are those propositions which follow directly from an axiom or a theorem. For instance, it follows from the axiom "there is only one line passing through two points" that "two lines can intersect at one point at most."

29. The content of a theorem. In any theorem one can distinguish two parts: the hypothesis and the conclusion. The **hypothesis** expresses what is considered given, the **conclusion** what is required to prove. For example, in the theorem "if central angles are congruent, then the corresponding arcs are congruent" the hypothesis is the first part of the theorem: "if central angles are congruent," and the conclusion is the second part: "then the corresponding arcs are congruent;" in other words, it is given (known to us) that the central angles are congruent, and it is required to prove that under this hypothesis the corresponding arcs are congruent.

The hypothesis and the conclusion of a theorem may sometimes consist of several separate hypotheses and conclusions; for instance, in the theorem "if a number is divisible by 2 and by 3, then it is divisible by 6," the hypothesis consists of two parts: "if a number is divisible by 2" and "if the number is divisible by 3."

It is useful to notice that any theorem can be rephrased in such a way that the hypothesis will begin with the word "if," and the conclusion with the word "then." For example, the theorem "vertical angles are congruent" can be rephrased this way: "*if* two angles are vertical, *then* they are congruent."

30. The converse theorem. The theorem converse to a given theorem is obtained by replacing the hypothesis of the given theorem with the conclusion (or some part of the conclusion), and the conclusion with the hypothesis (or some part of the hypothesis) of the given theorem. For instance, the following two theorems are converse to each other:

If central angles are congruent, then the corresponding arcs are congruent.

If arcs are congruent, then the corresponding central angles are congruent.

If we call one of these theorems **direct**, then the other one should be called **converse**.

In this example both theorems, the direct and the converse one, turn out to be true. This is not always the case. For example the theorem: "if two angles are vertical, then they are congruent" is true, but the converse statement: "if two angles are congruent, then they are vertical" is false.

Indeed, suppose that in some angle the bisector is drawn (Figure 13). It divides the angle into two smaller ones. These smaller angles are congruent to each other, but they are not vertical.

EXERCISES

42. Formulate definitions of supplementary angles (§22) and vertical angles (§26) using the notion of *sides* of an angle.

43. Find in the text the definitions of an angle, its vertex and sides, in terms of the notion of a *ray drawn from a point.*

*44.** In Introduction, find the definitions of a ray and a straight segment in terms of the notions of a *straight line* and a point. Are there definitions of a point, line, plane, surface, geometric solid? Why?
Remark: These are examples of geometric notions which are considered **undefinable.**

45. Is the following proposition from §6 a definition, axiom or theorem: "Two segments are congruent if they can be laid one onto the other so that their endpoints coincide"?

46. In the text, find the definitions of a geometric figure, and congruent geometric figures. Are there definitions of congruent segments, congruent arcs, congruent angles? Why?

47. Define a circle.

48. Formulate the proposition converse to the theorem: "If a number is divisible by 2 and by 3, then it is divisible by 6." Is the converse proposition true? Why?

49. In the proposition from §10: "Two arcs of the same circle are congruent if they can be aligned so that their endpoints coincide," separate the hypothesis from the conclusion, and state the converse proposition. Is the converse proposition true? Why?

50. In the theorem: "Bisectors of supplementary angles are perpendicular," separate the hypothesis from the conclusion, and formulate the converse proposition. Is the converse proposition true?

51. Give an example that disproves the proposition: "If the bisectors of two angles with a common vertex are perpendicular, then the angles are supplementary." Is the converse proposition true?

4 Polygons and triangles

31. Broken lines. Straight segments not lying on the same line are said to form a **broken line** (Figures 31, 32) if the endpoint of the first segment is the beginning of the second one, the endpoint of the second segment is the beginning of the third one, and so on. These segments are called **sides**, and the vertices of the angles formed by the adjacent segments **vertices** of the broken line. A broken line is denoted by the row of letters labeling its vertices and endpoints; for instance, one says: "the broken line $ABCDE$."

A broken line is called **convex** if it lies on one side of each of its segments continued indefinitely in both directions. For example, the broken line shown in Figure 31 is convex while the one shown in Figure 32 is not (it lies not on one side of the line BC).

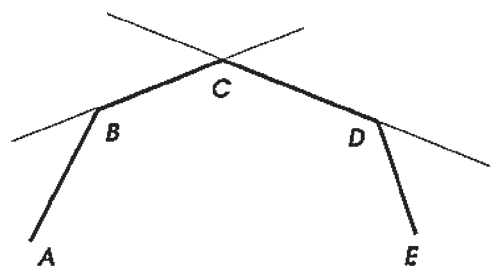

Figure 31

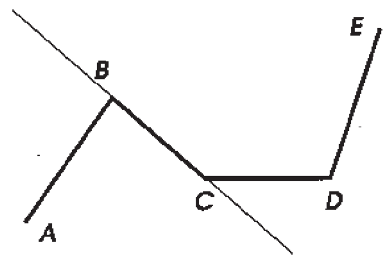

Figure 32

A broken line whose endpoints coincide is called **closed** (e.g. the lines $ABCDE$ or $ADCBE$ in Figure 33). A closed broken line may have self-intersections. For instance, in Figure 33, the line $ADCBE$ is self-intersecting, while $ABCDE$ is not.

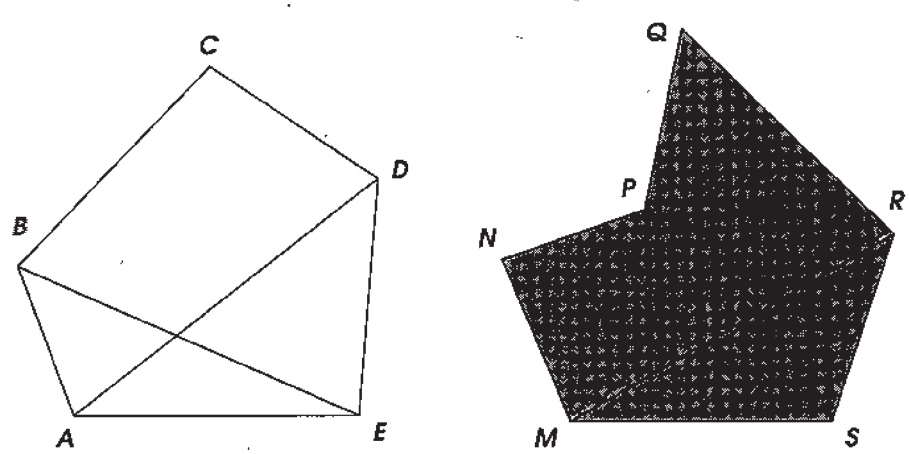

Figure 33

32. Polygons. The figure formed by a non-self-intersecting closed broken line together with the part of the plane bounded by this line is called a **polygon** (Figure 33). The sides and vertices of this broken line are called respectively **sides** and **vertices** of the polygon, and the angles formed by each two adjacent sides **(interior) angles** of the polygon. More precisely, the interior of a polygon's angle is considered that side which contains the interior part of the polygon in the vicinity of the vertex. For instance, the angle at the vertex P of the polygon $MNPQRS$ is the angle greater than $2d$ (with the interior region shaded in Figure 33). The broken line itself is called the **boundary** of the polygon, and the segment congruent to the sum of all of its sides — the **perimeter**. A half of the perimeter is often referred to as the **semiperimeter**.

A polygon is called **convex** if it is bounded by a convex broken line. For example, the polygon $ABCDE$ shown in Figure 33 is convex while the polygon $MNPQRS$ is not. We will mainly consider convex polygons.

Any segment (like AD, BE, MR, ..., Figure 33) which connects two vertices not belonging to the same side of a polygon is called a **diagonal** of the polygon.

The smallest number of sides in a polygon is three. Polygons are named according to the number of their sides: **triangles**, **quadrilaterals**, **pentagons**, **hexagons**, and so on.

The word "triangle" will often be replaced by the symbol $\triangle$.

33. Types of triangles. Triangles are classified by relative lengths of their sides and by the magnitude of their angles. With respect to the lengths of sides, triangles can be **scalene** (Figure 34) — when all three sides have different lengths, **isosceles** (Figure 35) — when two sides are congruent, and **equilateral** (Figure 36) — when all three sides are congruent.

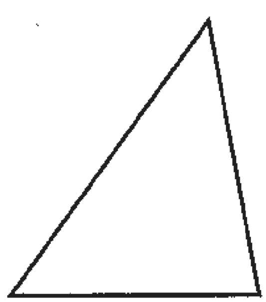

Figure 34

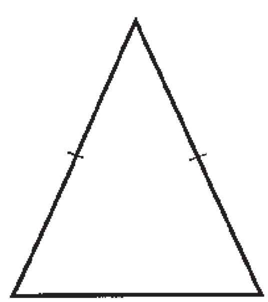

Figure 35

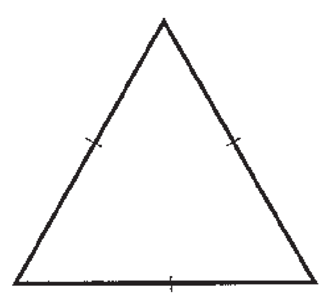

Figure 36

With respect to the magnitude of angles, triangles can be **acute** (Figure 34) — when all three angles are acute, **right** (Figure 37) —

when among the angles there is a right one, and **obtuse** (Figure 38) — when among the angles there is an obtuse one. [3]

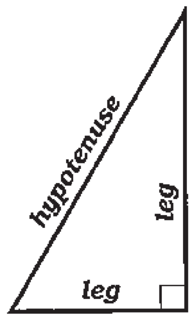

Figure 37

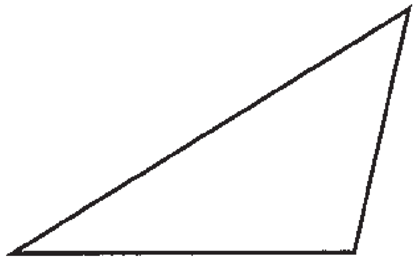

Figure 38

In a right triangle, the sides of the right angle are called **legs**, and the side opposite to the right angle the **hypotenuse**.

34. Important lines in a triangle. One of a triangle's sides is often referred to as **the base**, in which case the opposite vertex is called *the* vertex of the triangle, and the other two sides are called **lateral**. Then the perpendicular dropped from the vertex to the base or to its continuation is called an **altitude**. Thus, if in the triangle ABC (Figure 39), the side AC is taken for the base, then B is the vertex, and BD is the altitude.

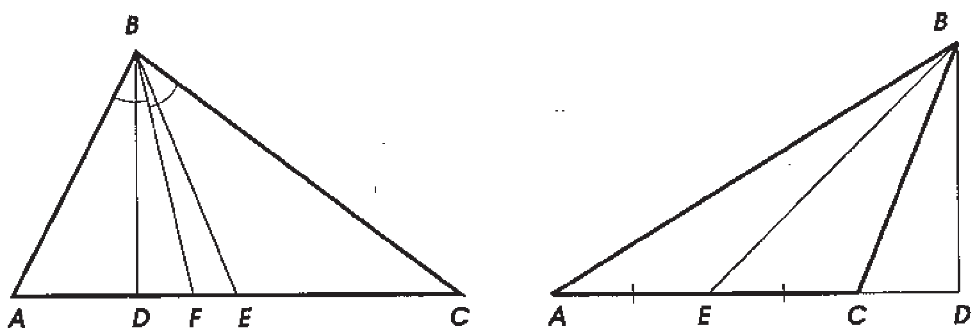

Figure 39

The segment (BE, Figure 39) connecting the vertex of a triangle with the midpoint of the base is called a **median**. The segment (BF) dividing the angle at the vertex into halves is called a **bisector** of the triangle (which generally speaking differs from both the median and the altitude).

[3] We will see in §43 that a triangle may have at most one right or obtuse angle.

Any triangle has three altitudes, three medians, and three bisectors, since each side of the triangle can take on the role of the base.

In an isosceles triangle, usually the side other than each of the two congruent ones is called the base. Respectively, the vertex of an isosceles triangle is the vertex of that angle which is formed by the congruent sides.

EXERCISES

52. Four points on the plane are vertices of three different quadrilaterals. How can this happen?

53. Can a convex broken line self-intersect?

54. Is it possible to tile the entire plane by non-overlapping polygons all of whose angles contain 140° each?

55. Prove that each diagonal of a quadrilateral either lies entirely in its interior, or entirely in its exterior. Give an example of a pentagon for which this is false.

56. Prove that a closed convex broken line is the boundary of a polygon.

57. Is an equilateral triangle considered isosceles? Is an isosceles triangle considered scalene?

*58.** How many intersection points can three straight lines have?

59. Prove that in a right triangle, three altitudes pass through a common point.

60. Show that in any triangle, every two medians intersect. Is the same true for every two bisectors? altitudes?

61. Give an example of a triangle such that only one of its altitudes lies in its interior.

5 Isosceles triangles and symmetry

35. Theorems.

(1) ***In an isosceles triangle, the bisector of the angle at the vertex is at the same time the median and the altitude.***

(2) ***In an isosceles triangle, the angles at the base are congruent.***

Let $\triangle ABC$ (Figure 40) be isosceles, and let the line BD be the bisector of the angle B at the vertex of the triangle. It is required to prove that this bisector BD is also the median and the altitude.

Imagine that the diagram is folded along the line BD so that $\angle ABD$ falls onto $\angle CBD$. Then, due to congruence of the angles 1 and 2, the side AB will fall onto the side CB, and due to congruence of these sides, the point A will merge with C. Therefore DA will coincide with DC, the angle 3 will coincide with the angle 4, and the angle 5 with 6. Therefore

$$DA = DC, \quad \angle 3 = \angle 4, \quad \text{and} \quad \angle 5 = \angle 6.$$

It follows from $DA = DC$ that BD is the median. It follows from the congruence of the angles 3 and 4 that these angles are right, and hence BD is the altitude of the triangle. Finally, the angles 5 and 6 at the base of the triangle are congruent.

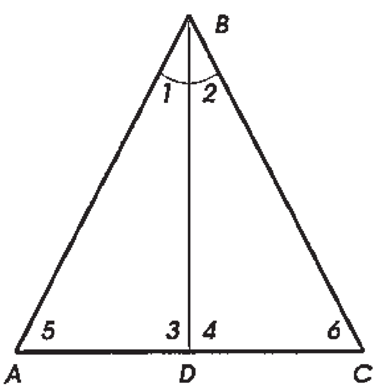

Figure 40

36. Corollary. We see that in the isosceles triangle ABC (Figure 40) the very same line BD possesses four properties: it is the bisector drawn from the vertex, the median to the base, the altitude dropped from the vertex to the base, and finally the perpendicular erected from the base at its midpoint.

Since each of these properties determines the position of the line BD unambiguously, then the validity of any of them implies all the others. For example, *the altitude dropped to the base of an isosceles triangle is at the same time its bisector drawn from the vertex, the median to the base, and the perpendicular erected at its midpoint.*

37. Axial symmetry. If two points (A and A', Figure 41) are situated on the opposite sides of a line a, on the same perpendicular to this line, and the same distance away from the foot of the perpendicular (i.e. if AF is congruent to FA'), then such points are called **symmetric** about the line a.

Two figures (or two parts of the same figure) are called symmetric about a line if for each point of one figure (A, B, C, D, E, ..., Figure 41) the point symmetric to it about this line (A', B', C', D', E', ...) belongs to the other figure, and *vice versa*. A figure is said to have an **axis of symmetry** a if this figure is symmetric to itself about the line a, i.e. if for any point of the figure the symmetric point also belongs to the figure.

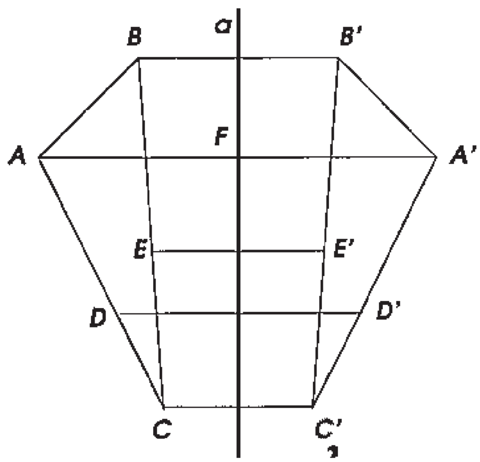

Figure 41

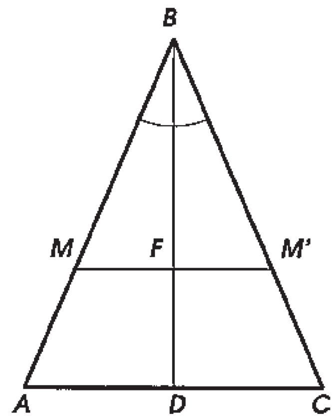

Figure 42

For example, we have seen that the isosceles triangle ABC (Figure 42) is divided by the bisector BD into two triangles (left and right) which can be identified with each other by folding the diagram along the bisector. One can conclude from this that whatever point is taken on the left half of the isosceles triangle, one can always find the point symmetric to it in the right half. For instance, on the side AB, take a point M. Mark on the side BC the segment BM' congruent to BM. We obtain the point M' in the triangle symmetric to M about the axis BD. Indeed, $\triangle MBM'$ is isosceles since $BM = BM'$. Let F denote the intersection point of the segment MM' with the bisector BD of the angle B. Then BK is the bisector in the isosceles triangle MBM'. By §35 it is also the altitude and the median. Therefore MM' is perpendicular to BD, and $MF = M'F$, i.e. M and M' are situated on the opposite sides of BD, on the same perpendicular to BD, and the same distance away from its foot F. Thus ***in an isosceles triangle, the bisector of the angle at the vertex is an axis of symmetry of the triangle.***

38. Remarks. (1) Two symmetric figures can be superimposed by rotating one of them in space about the axis of symmetry until the rotated figure falls into the original plane again. Conversely, if two figures can be identified with each other by turning the plane in space about a line lying in the plane, then these two figures are symmetric about this line.

(2) Although symmetric figures can be superimposed, they are not identical in their position in the plane. This should be understood in the following sense: in order to superimpose two symmetric figures it is *necessary* to flip one of them around and therefore to pull it off the plane temporarily; if however a figure is bound to remain in the plane, no motion can generally speaking identify it with the figure symmetric to it about a line. For example, Figure 43 shows two pairs of symmetric letters: "b" and "d," and "p" and "q." By rotating the letters inside the page one can transform "b" into "q," and "d" into "p," but it is impossible to identify "b" or "q" with "d" or "p" without lifting the symbols off the page.

(3) Axial symmetry is frequently found in nature (Figure 44).

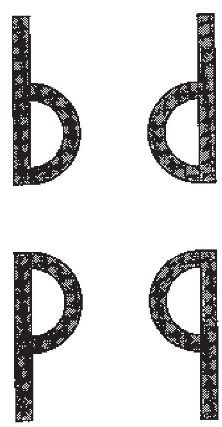

Figure 43

Figure 44

EXERCISES

62. How many axes of symmetry does an equilateral triangle have? How about an isosceles triangle which is not equilateral?

63.* How many axes of symmetry can a quadrilateral have?

64. A **kite** is a quadrilateral symmetric about a diagonal. Give an example of: (a) a kite; (b) a quadrilateral which is not a kite but has an axis of symmetry.

65. Can a pentagon have an axis of symmetry passing through two (one, none) of its vertices?

66.* Two points A and B are given on the same side of a line MN.

Find a point C on MN such that the line MN would make congruent angles with the sides of the broken line ACB.

Prove theorems:

67. In an isosceles triangle, two medians are congruent, two bisectors are congruent, two altitudes are congruent.

68. If from the midpoint of each of the congruent sides of an isosceles triangle, the segment perpendicular to this side is erected and continued to its intersection with the other of the congruent sides of the triangle, then these two segments are congruent.

69. A line perpendicular to the bisector of an angle cuts off congruent segments on its sides.

70. An equilateral triangle is **equiangular** (i.e. all of its angles are congruent).

71. Vertical angles are symmetric to each other with respect to the bisector of their supplementary angles.

72. A triangle that has two axes of symmetry has three axes of symmetry.

73. A quadrilateral is a kite if it has an axis of symmetry passing through a vertex.

74. Diagonals of a kite are perpendicular.

6 Congruence tests for triangles

39. Preliminaries. As we know, two geometric figures are called congruent if they can be identified with each other by superimposing. Of course, in the identified triangles, all their corresponding elements, such as sides, angles, altitudes, medians and bisectors, are congruent. However, in order to ascertain that two triangles are congruent, there is no need to establish congruence of all their corresponding elements. It suffices only to verify congruence of some of them.

40. Theorems. [4]

(1) **SAS-test:** ***If two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in another triangle, then such triangles are congruent.***

(2) **ASA-test:** ***If one side and two angles adjacent to it in one triangle are congruent respectively to one side and two***

[4] SAS stands for "side–angle–side", ASA for "angle-side-angle, and of course SSS for "side-side-side."

angles adjacent to it in another triangle, then such triangles are congruent.

(3) **SSS-test:** ***If three sides of one triangle are congruent respectively to three sides of another triangle, then such triangles are congruent.***

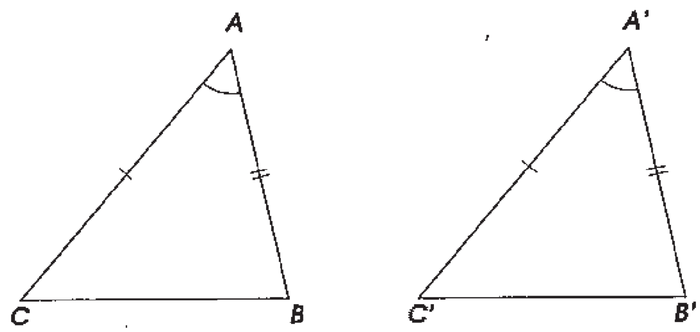

Figure 45

(1) Let ABC and $A'B'C'$ be two triangles (Figure 45) such that

$$AC = A'C', \quad AB = A'B', \quad \angle A = \angle A'.$$

It is required to prove that these triangles are congruent.

Superimpose $\triangle ABC$ onto $\triangle A'B'C'$ in such a way that A would coincide with A', the side AC would go along $A'C'$, and the side AB would lie on the same side of $A'C'$ as $A'B'$. [5] Then: since AC is congruent to $A'C'$, the point C will merge with C'; due to congruence of $\angle A$ and $\angle A'$, the side AB will go along $A'B'$, and due to congruence of these sides, the point B will merge with B'. Therefore the side BC will coincide with $B'C'$ (since two points can be joined by only one line), and hence the entire triangles will be identified with each other. Thus they are congruent.

(2) Let ABC and $A'B'C'$ (Figure 46) be two triangles such that

$$\angle C = \angle C', \quad \angle B = \angle B', \quad CB = C'B'.$$

It is required to prove that these triangles are congruent. Superimpose $\triangle ABC$ onto $\triangle A'B'C'$ in such a way that the point C would coincide with C', the side CB would go along $C'B'$, and the vertex A would lie on the same side of $C'B'$ as A'. Then: since CB is congruent to $C'B'$, the point B will merge with B', and due to congruence of

[5]For this and some other operations in this section it might be necessary to flip the triangle over.

the angles B and B', and C and C', the side BA will go along $B'A'$, and the side CA will go along $C'A'$. Since two lines can intersect only at one point, the vertex A will have to merge with A'. Thus the triangles are identified and are therefore congruent.

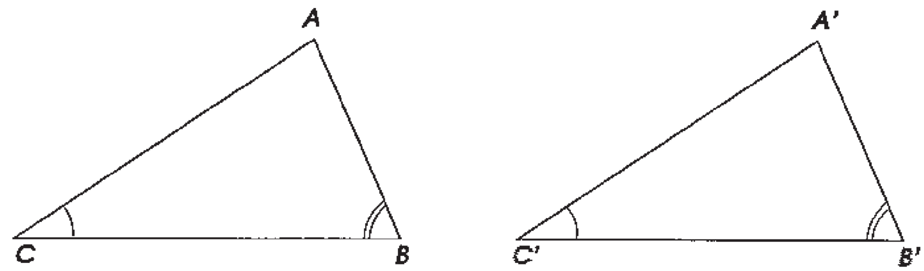

Figure 46

(3) Let ABC and $A'B'C'$ be two triangles such that

$$AB = A'B', \quad BC = B'C', \quad CA = C'A'.$$

It is required to prove that these triangles are congruent. Proving this test by superimposing, the same way as we proved the first two tests, turns out to be awkward, because knowing nothing about the measure of the angles, we would not be able to conclude from coincidence of two corresponding sides that the other sides coincide as well. Instead of superimposing, let us apply *juxtaposing*.

Juxtapose $\triangle ABC$ and $\triangle A'B'C'$ in such a way that their congruent sides AC and $A'C'$ would coincide (i.e. A would merge with A' and C with C'), and the vertices B and B' would lie on the opposite sides of $A'C'$. Then $\triangle ABC$ will occupy the position $\triangle A'B''C'$ (Figure 47). Joining the vertices B' and B'' we obtain two isosceles triangles $B'A'B''$ and $B'C'B''$ with the common base $B'B''$. But in an isosceles triangle, the angles at the base are congruent (§35). Therefore $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, and hence $\angle A'B'C' = \angle A'B''C' = \angle B$. But then the given triangles must be congruent, since two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in the other triangle.

Remark. In congruent triangles, congruent angles are opposed to congruent sides, and conversely, congruent sides are opposed to congruent angles.

The congruence tests just proved, and the skill of recognizing congruent triangles by the above criteria facilitate solutions to many geometry problems and are necessary in the proofs of many theorems. These congruence tests are the principal means in discovering properties of complex geometric figures. The reader will have many occasions to see this.

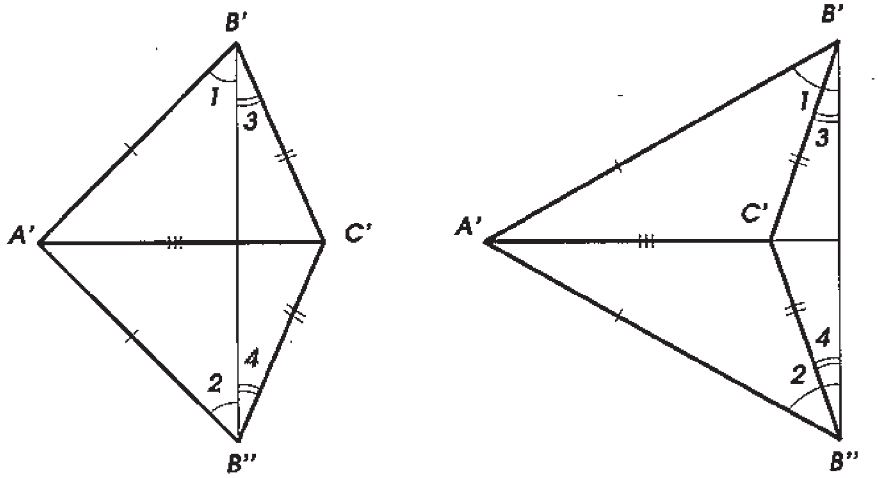

Figure 47

EXERCISES

75. Prove that a triangle that has two congruent angles is isosceles.

76. In a given triangle, an altitude is a bisector. Prove that the triangle is isosceles.

77. In a given triangle, an altitude is a median. Prove that the triangle is isosceles.

78. On each side of an equilateral triangle ABC, congruent segments AB', BC', and AC' are marked, and the points A', B', and C' are connected by lines. Prove that the triangle $A'B'C'$ is also equilateral.

79. Suppose that an angle, its bisector, and one side of this angle in one triangle are respectively congruent to an angle, its bisector, and one side of this angle in another triangle. Prove that such triangles are congruent.

80. Prove that if two sides and the median drawn to the first of them in one triangle are respectively congruent to two sides and the median drawn to the first of them in another triangle, then such triangles are congruent.

81. Give an example of two non-congruent triangles such that two sides and one angle of one triangle are respectively congruent to two sides and one angle of the other triangle.

*82.** On one side of an angle A, the segments AB and AC are marked, and on the other side the segments $AB' = AB$ and $AC' = AC$. Prove that the lines BC' and $B'C$ meet on the bisector of the angle A.

83. Derive from the previous problem a method of constructing the bisector using straightedge and compass.

84. Prove that in a convex pentagon: (a) if all sides are congruent, and all diagonals are congruent, then all interior angles are congruent, and (b) if all sides are congruent, and all interior angles are congruent, then all diagonals are congruent.

85. Is this true that in a convex polygon, if all diagonals are congruent, and all interior angles are congruent, then all sides are congruent?

7 Inequalities in triangles

41. Exterior angles. The angle supplementary to an angle of a triangle (or polygon) is called an **exterior angle** of this triangle (polygon).

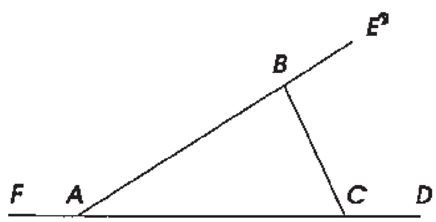

Figure 48

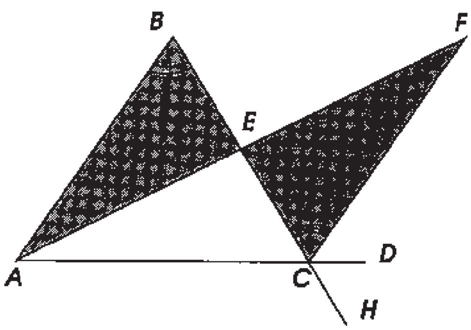

Figure 49

For instance (Figure 48), $\angle BCD$, $\angle CBE$, $\angle BAF$ are exterior angles of the triangle ABC. In contrast with the exterior angles, the angles of the triangle (polygon) are sometimes called **interior.**

For each interior angle of a triangle (or polygon), one can construct two exterior angles (by extending one or the other side of the angle). Such two exterior angles are congruent since they are vertical.

42. Theorem. ***An exterior angle of a triangle is greater than each interior angle not supplementary to it.***

For example, let us prove that the exterior angle BCD of $\triangle ABC$ (Figure 49) is greater than each of the interior angles A and B not supplementary to it.

Through the midpoint E of the side BC, draw the median AE and on the continuation of the median mark the segment EF congruent to AE. The point F will obviously lie in the interior of the angle BCD. Connect F with C by a segment. The triangles ABE and EFC (shaded in Figure 49) are congruent since at the vertex E they have congruent angles enclosed between two respectively congruent sides. From congruence of the triangles we conclude that the angles B and ECF, opposite to the congruent sides AE and EF, are congruent too. But the angle ECF forms a part of the exterior angle BCD and is therefore smaller than $\angle BCD$. Thus the angle B is smaller than the angle BCD.

By continuing the side BC past the point C we obtain the exterior angle ACH congruent to the angle BCD. If from the vertex B, we draw the median to the side AC and double the median by continuing it past the side AC, then we will similarly prove that the angle A is smaller than the angle ACH, i.e. it is smaller than the angle BCD.

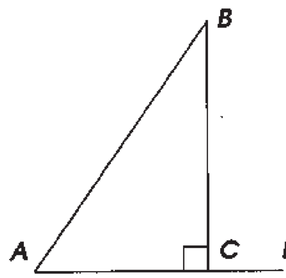

Figure 50

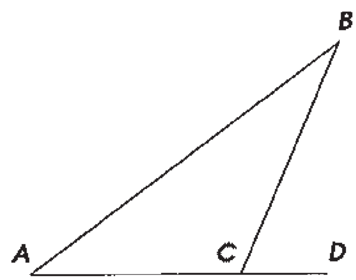

Figure 51

43. Corollary. *If in a triangle one angle is right or obtuse, then the other two angles are acute.*

Indeed, suppose that the angle C in $\triangle ABC$ (Figure 50 or 51) is right or obtuse. Then the supplementary to it exterior angle BCD has to be right or acute. Therefore the angles A and B, which by the theorem are smaller than this exterior angle, must both be acute.

44. Relationships between sides and angles of a triangle.

Theorems. ***In any triangle***

(1) ***the angles opposite to congruent sides are congruent;***

(2) ***the angle opposite to a greater side is greater.***

(1) If two sides of a triangle are congruent, then the triangle is isosceles, and therefore the angles opposite to these sides have to be congruent as the angles at the base of an isosceles triangle (§35).

(2) Let in $\triangle ABC$ (Figure 52) the side AB be greater than BC. It is required to prove that the angle C is greater than the angle A.

On the greater side BA, mark the segment BD congruent to the smaller side BC and draw the line joining D with C. We obtain an isosceles triangle DBC, which has congruent angles at the base, i.e. $\angle BDC = \angle BCD$. But the angle BDC, being an exterior angle with respect to $\triangle ADC$, is greater than the angle A, and hence the angle BCD is also greater than the angle A. Therefore the angle BCA containing $\angle BCD$ as its part is greater than the angle A too.

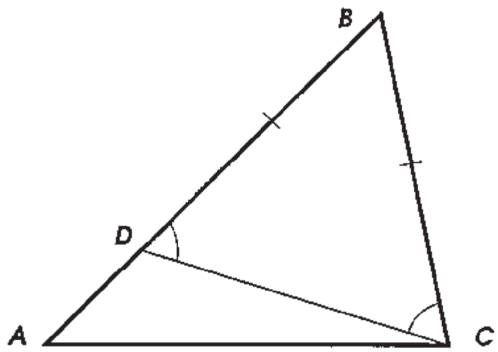

Figure 52

45. The converse theorems. ***In any triangle***

(1) ***the sides opposite to congruent angles are congruent;***

(2) ***the side opposite to a greater angle is greater.***

(1) Let in $\triangle ABC$ the angles A and C be congruent (Figure 53); it is required to prove that $AB = BC$.

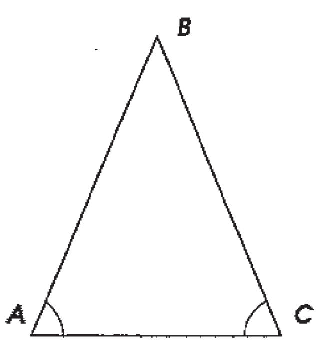

Figure 53

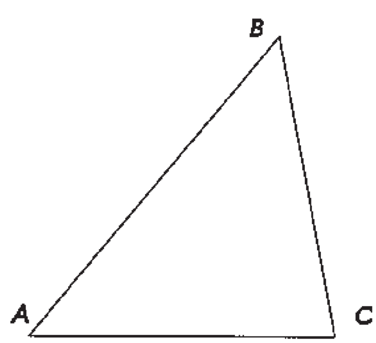

Figure 54

Suppose the contrary is true, i.e. that the sides AB and BC are not congruent. Then one of these sides is greater than the other, and therefore according to the direct theorem, one of the angles A and C has to be greater than the other. But this contradicts the hypothesis that $\angle A = \angle C$. Thus the assumption that AB and BC are non-congruent is impossible. This leaves only the possibility that $AB = BC$.

(2) Let in $\triangle ABC$ (Figure 54) the angle C be greater than the angle A. It is required to prove that $AB > BC$.

Suppose the contrary is true, i.e. that AB is not greater than BC. Then two cases can occur: either $AB = BC$ or $AB < BC$.

According to the direct theorem, in the first case the angle C would have been congruent to the angle A, and in the second case the angle C would have been smaller than the angle A. Either conclusion contradicts the hypothesis, and therefore both cases are excluded. Thus the only remaining possibility is $AB > BC$.

Corollary.

(1) *In an equilateral triangle all angles are congruent.*

(2) *In an equiangular triangle all sides are congruent.*

46. Proof by contradiction. The method we have just used to prove the converse theorems is called **proof by contradiction**, or **reductio ad absurdum**. In the beginning of the argument the assumption contrary to what is required to prove is made. Then by reasoning on the basis of this assumption one arrives at a contradiction (absurd). This result forces one to reject the initial assumption and thus to accept the one that was required to prove. This way of reasoning is frequently used in mathematical proofs.

47. A remark on converse theorems. It is a mistake, not uncommon for beginning geometry students, to assume that the converse theorem is automatically established whenever the validity of a direct theorem has been verified. Hence the false impression that proof of converse theorems is unnecessary at all. As it can be shown by examples, like the one given in §30, this conclusion is erroneous. Therefore converse theorems, when they are valid, require separate proofs.

However, in the case of congruence or non-congruence of two sides of a triangle ABC, e.g. the sides AB and BC, only the following three cases can occur:

$$AB = BC, \quad AB > BC, \quad AB < BC.$$

Each of these three cases excludes the other two: say, if the first case $AB = BC$ takes place, then neither the 2nd nor the 3rd case is possible. In the theorem of §44, we have considered all the three cases and arrived at the following respective conclusions regarding the opposite angles C and A:

$$\angle C = \angle A, \quad \angle C > \angle A, \quad \angle C < \angle A.$$

Each of these conclusions excludes the other two. We have also seen in §45 that the converse theorems are true and can be easily proved by *reductio ad absurdum.*

In general, if in a theorem, or several theorems, we address all possible mutually exclusive cases (which can occur regarding the magnitude of a certain quantity or disposition of certain parts of a figure), and it turns out that in these cases we arrive at mutually exclusive conclusions (regarding some other quantities or parts of the figure), then we can claim *a priori* that the converse propositions also hold true.

We will encounter this rule of convertibility quite often.

48. Theorem. ***In a triangle, each side is smaller than the sum of the other two sides.***

If we take a side which is not the greatest one in a triangle, then of course it will be smaller than the sum of the other two sides. Therefore we need to prove that even the greatest side of a triangle is smaller than the sum of the other two sides.

In $\triangle ABC$ (Figure 55), let the greatest side be AC. Continuing the side AB past B mark on it the segment $BD = BC$ and draw DC. Since $\triangle BDC$ is isosceles, then $\angle D = \angle DCB$. Therefore the angle D is smaller than the angle DCA, and hence in $\triangle ADC$ the side AC is smaller than AD (§45), i.e. $AC < AB + BD$. Replacing BD with BC we get

$$AC < AB + BC.$$

Corollary. From both sides of the obtained inequality, subtract AB or BC:

$$AC - AB < BC, \quad AC - BC < AB.$$

Reading these inequalities from right to left we see that each of the sides BC and AB is greater than the difference of the other two sides. Obviously, the same can also be said about the greatest side AC, and therefore *in a triangle, each side is greater than the difference of the other two sides.*

Remarks. (1) The inequality described in the theorem is often called the **triangle inequality**.

(2) When the point B lies on the segment AC, the triangle inequality turns into the equality $AC = AB + BC$. More generally, if three points lie on the same line (and thus do not form a triangle), then the greatest of the three segments connecting these points is the sum of the other two segments. Therefore ***for any three points*** it is

still true that ***the segment connecting two of them is smaller than or congruent to the sum of the other two segments.***

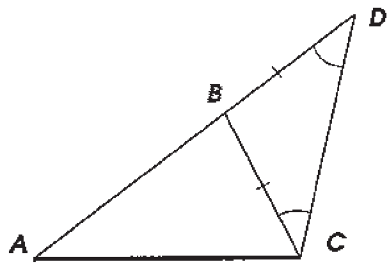

Figure 55

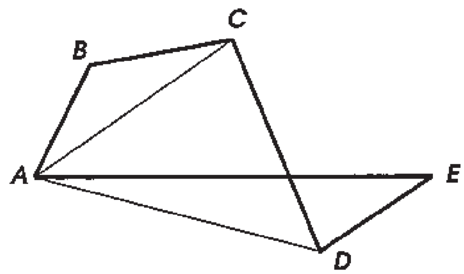

Figure 56

49. Theorem. ***The line segment connecting any two points is smaller than any broken line connecting these points.***

If the broken line in question consists of only two sides, then the theorem has already been proved in §48. Consider the case when the broken line consists of more than two sides. Let AE (Figure 56) be the line segment connecting the points A and E, and let $ABCDE$ be a broken line connecting the same points. We are required to prove that AE is smaller than the sum $AB + BC + CD + DE$.

Connecting A with C and D and using the triangle inequality we find:

$$AE \leq AD + DE, \quad AD \leq AC + CD, \quad AC \leq AB + BC.$$

Moreover, these inequalities cannot turn into equalities all at once. Indeed, if this happened, then (Figure 57) D would lie on the segment AE, C on AD, B on AB, i.e. $ABCDE$ would not be a broken line, but the straight segment AE. Thus adding the inequalities termwise

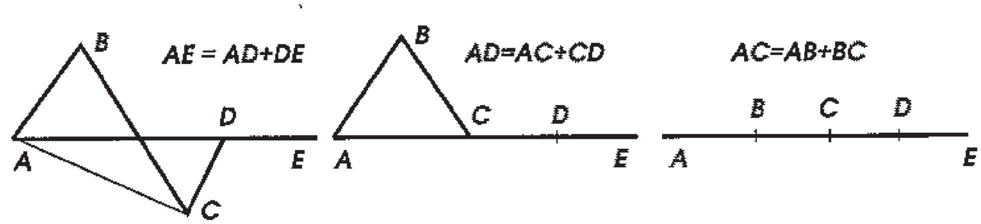

Figure 57

and subtracting AD and AC from both sides we get

$$AE < AB + BC + CD + DE.$$

50. Theorem. ***If two sides of one triangle are congruent respectively to two sides of another triangle, then:***

(1) ***the greater angle contained by these sides is opposed to the greater side;***

(2) *vice versa,* ***the greater of the non-congruent sides is opposed to the greater angle.***

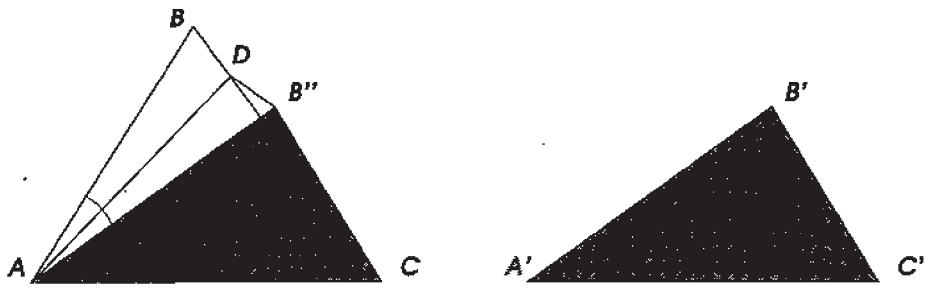

Figure 58

(1) In $\triangle ABC$ and $\triangle A'B'C'$, we are given:

$$AB = A'B', \quad AC = A'C', \quad \angle A > \angle A'.$$

We are required to prove that $BC > B'C'$. Put $\triangle A'B'C'$ onto $\triangle ABC$ in a way (shown in Figure 58) such that the side $A'C'$ would coincide with AC. Since $\angle A' < \angle A$, then the side $A'B'$ will lie inside the angle A. Let $\triangle A'B'C'$ occupy the position $AB''C$ (the vertex B'' may fall outside or inside of $\triangle ABC$, or on the side BC, but the forthcoming argument applies to all these cases). Draw the bisector AD of the angle BAB'' and connect D with B''. Then we obtain two triangles ABD and DAB'' which are congruent because they have a common side AD, $AB = AB''$ by hypothesis, and $\angle BAD = \angle BAD''$ by construction. Congruence of the triangles implies $BD = DB''$. From $\triangle DCB''$ we now derive: $B''C < B''D + DC$ (§48). Replacing $B''D$ with BD we get

$$B''C < BD + DC, \quad \text{and hence} \quad B'C' < BC.$$

(2) Suppose in the same triangles ABC and $A'B'C'$ we are given that $AB = A'B'$, $AC = A'C'$ and $BC > B'C'$; let us prove that $\angle A > \angle A'$.

Assume the contrary, i.e. that the $\angle A$ is not greater than $\angle A'$. Then two cases can occur: either $\angle A = \angle A'$ or $\angle A < \angle A'$. In the first case the triangles would have been congruent (by the SAS-test) and therefore the side BC would have been congruent to $B'C'$, which contradicts the hypotheses. In the second case the side BC would have been smaller than $B'C'$ by part (1) of the theorem, which contradicts the hypotheses too. Thus both of these cases are excluded; the only case that remains possible is $\angle A > \angle A'$.

EXERCISES

86. Can an exterior angle of an isosceles triangle be smaller than the supplementary interior angle? Consider the cases when the angle is: (a) at the base, and (b) at the vertex.

87. Can a triangle have sides: (a) 1, 2, and 3 *cm* (centimeters) long? (b) 2, 3, and 4 *cm* long?

88. Can a quadrilateral have sides: 2, 3, 4, and 10 *cm* long?

Prove theorems:

89. A side of a triangle is smaller than its semiperimeter.

90. A median of a triangle is smaller than its semiperimeter.

91.* A median drawn to a side of a triangle is smaller than the semisum of the other two sides.
Hint: Double the median by prolonging it past the midpoint of the first side.

92. The sum of the medians of a triangle is smaller than its perimeter but greater than its semi-perimeter.

93. The sum of the diagonals of a quadrilateral is smaller than its perimeter but greater than its semi-perimeter.

94. The sum of segments connecting a point inside a triangle with its vertices is smaller than the semiperimeter of the triangle.

95.* Given an acute angle XOY and an interior point A. Find a point B on the side OX and a point C on the side OY such that the perimeter of the triangle ABC is minimal.
Hint: Introduce points symmetric to A with respect to the sides of the angle.

8 Right triangles

51. Comparative length of the perpendicular and a slant.

Theorem. ***The perpendicular dropped from any point to a line is smaller than any slant drawn from the same point to this line.***

Let AB (Figure 59) be the perpendicular dropped from a point A to the line MN, and AC be any slant drawn from the same point A to the line MN. It is required to show that $AB < AC$.

In $\triangle ABC$, the angle B is right, and the angle C is acute (§43). Therefore $\angle C < \angle B$, and hence $AB < AC$, as required.

Remark. By "the distance from a point to a line," one means the *shortest* distance which is measured along the perpendicular dropped from this point to the line.

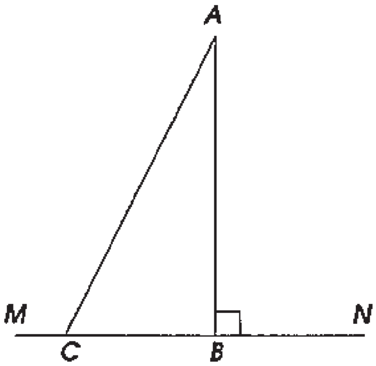

Figure 59

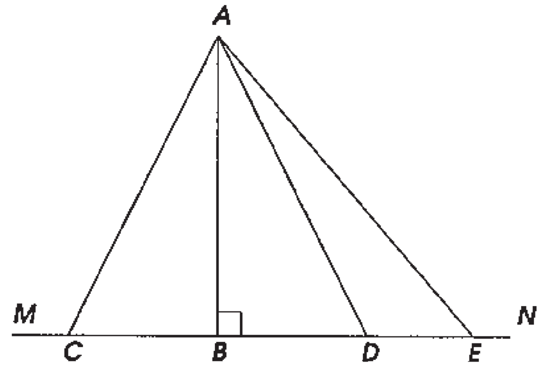

Figure 60

52. Theorem. ***If the perpendicular and some slants are drawn to a line from the same point outside this line, then:***

(1) ***if the feet of the slants are the same distance away from the foot of the perpendicular, then such slants are congruent;***

(2) ***if the feet of two slants are not the same distance away from the foot of the perpendicular, then the slant whose foot is farther away from the foot of the perpendicular is greater.***

(1) Let AC and AD (Figure 60) be two slants drawn from a point A to the line MN and such that their feet C and D are the same distance away from the foot B of the perpendicular AB, i.e. $CB = BD$. It is required to prove that $AC = AD$.

In the triangles ABC and ABD, AB is a common side, and beside this $BC = BD$ (by hypothesis) and $\angle ABC = \angle ABD$ (as right angles). Therefore these triangles are congruent, and thus $AC = AD$.

(2) Let AC and AE (Figure 59) be two slants drawn from the point A to the line MN and such that their feet are not the same distance away from the foot of the perpendicular; for instance, let $BE > BC$. It is required to prove that $AE > AC$.

Mark $BD = BC$ and draw AD. By part (1), $AD = AC$. Compare AE with AD. The angle ADE is exterior with respect to $\triangle ABD$ and therefore it is greater than the right angle. Therefore the angle ADE is obtuse, and hence the angle AED must be acute (§43). It follows that $\angle ADE > \angle AED$, therefore $AE > AD$, and thus $AE > AC$.

53. The converse theorems. ***If some slants and the perpendicular are drawn to a line from the same point outside this line, then:***

(1) ***if two slants are congruent, then their feet are the same distance away from the foot of the perpendicular;***

(2) ***if two slants are not congruent, then the foot of the greater one is farther away from the foot of the perpendicular.***

We leave it to the readers to prove these theorems (by the method of *reductio ad absurdum*).

54. Congruence tests for right triangles. Since in right triangles the angles contained by the legs are always congruent as right angles, then ***right triangles are congruent:***

(1) *if the legs of one of them are congruent respectively to the legs of the other;*

(2) *if a leg and the acute angle adjacent to it in one triangle are congruent respectively to a leg and the acute angle adjacent to it in the other triangle.*

These two tests require no special proof, since they are particular cases of the general SAS- and ASA-tests. Let us prove the following two tests which apply to right triangles only.

55. Two tests requiring special proofs.

Theorems. ***Two right triangles are congruent:***

(1) ***if the hypotenuse and an acute angle of one triangle are congruent to respectively the hypotenuse and an acute angle of the other.***

(2) ***if the hypotenuse and a leg of one triangle are congruent respectively to the hypotenuse and a leg of the other.***

(1) Let ABC and $A_1B_1C_1$ (Figure 61) be two right triangles such that $AB = A_1B_1$ and $\angle A = \angle A_1$. It is required to prove that these triangles are congruent.

Put $\triangle ABC$ onto $\triangle A_1B_1C_1$ in a way such that their congruent hypotenuses coincide. By congruence of the angles A and A_1, the leg AC will go along A_1C_1. Then, if we assume that the point C

occupies a position C_2 or C_3 different from C_1, we will have two perpendiculars (B_1C_1 and B_1C_2, or B_1C_1 and B_1C_3) dropped from the same point B' to the line $A'C'$. Since this is impossible (§24), we conclude that the point C will merge with C_1.

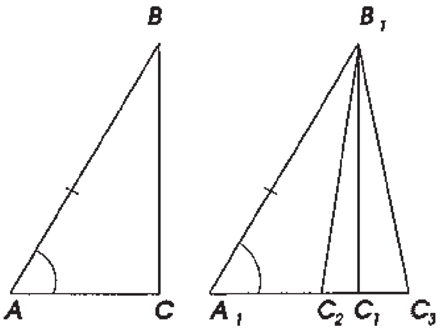

Figure 61

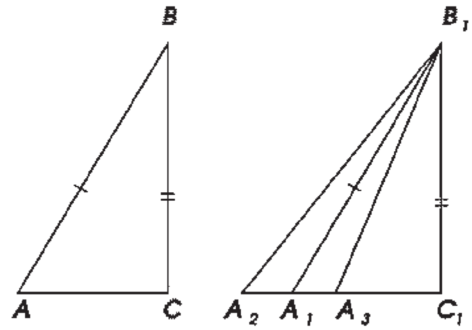

Figure 62

(2) Let (Figure 62), in the right triangles, it be given: $AB = A_1B_1$ and $BC = B_1C_1$. It is required to prove that the triangles are congruent. Put ΔABC onto $\Delta A_1B_1C_1$ in a way such that their congruent legs BC and B_1C_1 coincide. By congruence of right angles, the side CA will go along C_1A_1. Then, if we assume that the hypotenuse AB occupies a position A_2B_1 or A_3B_1 different from A_1B_1, we will have two congruent slants (A_1B_1 and A_2B_1, or A_1B_1 and A_3B_1) whose feet are not the same distance away from the foot of the perpendicular B_1C_1. Since this is impossible (§53) we conclude that AB will be identified with A_1B_1.

EXERCISES

Prove theorems:

96. Each leg of a right triangle is smaller than the hypotenuse.

97. A right triangle can have at most one axis of symmetry.

98. At most two congruent slants to a given line can be drawn from a given point.

99.* Two isosceles triangles with a common vertex and congruent lateral sides cannot fit one inside the other.

100. The bisector of an angle is its axis of symmetry.

101. A triangle is isosceles if two of its altitudes are congruent.

102. A median in a triangle is equidistant from the two vertices not lying on it.

103.* A line and a circle can have at most two common points.

9 Segment and angle bisectors

56. The perpendicular bisector, i.e. the perpendicular to a segment erected at the midpoint of the segment, and the bisector of an angle have very similar properties. To see the resemblance better we will describe the properties in a parallel fashion.

(1) ***If a point*** (K, Figure 63) ***lies on the perpendicular*** (MN) ***erected at the midpoint of a segment*** (AB), ***then the point is the same distance away from the endpoints of the segment*** (i.e. $KA = KB$).

Since $MN \perp AB$ and $AO = OB$, AK and KB are slants to AB, and their feet are the same distance away from the foot of the perpendicular. Therefore $KA = KB$.

(1) ***If a point*** (K, Figure 64) ***lies on the bisector*** (OM) ***of an angle*** (AOB), ***then the point is the same distance away from the sides of the angle*** (i.e. the perpendiculars KD and KC are congruent).

Since OM bisects the angle, the right triangles OCK and ODK are congruent, as they have the common hypotenuse and congruent acute angles at the vertex O. Therefore $KC = KD$.

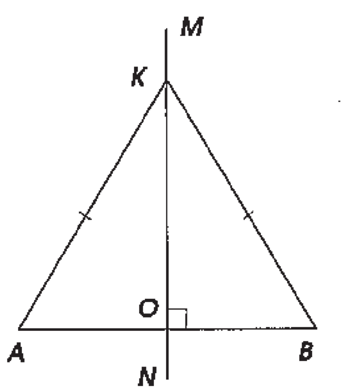

Figure 63

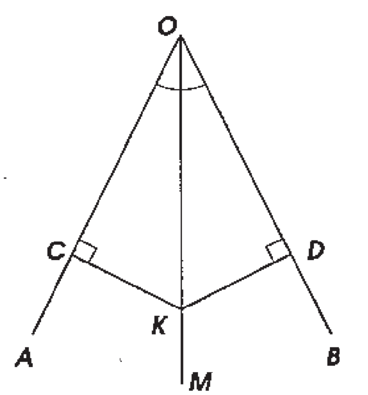

Figure 64

(2) The converse theorem. ***If a point*** (K, Figure 63) ***is the same distance away from the endpoints of the segment*** AB (i.e. if $KA = KB$), ***then the point lies on the perpendicular to*** AB ***passing through its midpoint.***

(2) The converse theorem. ***If an interior point of an angle*** (K, Figure 64) ***is the same distance away from its sides*** (i.e. if the perpendiculars KC and KD are congruent) ***then it lies on the bisector of this angle.***

Through K, draw the line $MN \perp AB$. We get two right triangles KAO and KBO which are congruent as having congruent hypotenuses and the common leg KO. Therefore the line MN drawn through K to be perpendicular to AB bisects it.

Through O and K, draw the line OM. Then we get two right triangles OCK and ODK which are congruent as having the common hypotenuse and the congruent legs CK and DK. Hence they have congruent angles at the vertex O, and therefore the line OM drawn to pass through K bisects the angle AOB.

57. Corollary. From the two proven theorems (direct and converse) one can also derive the following theorems:

If a point does not lie on the perpendicular erected at the midpoint of a segment then the point is unequal distances away from the endpoints of this segment.

If an interior point of an angle does not lie on the ray bisecting it, then the point is unequal distances away from the sides of this angle.

We leave it to the readers to prove these theorems (using the method *reductio ad absurdum*).

58. Geometric locus. The **geometric locus** of points satisfying a certain condition is the curve (or the surface in the space) or, more generally, the set of points, which contains all the points satisfying this condition and contains no points which do not satisfy it.

For instance, the geometric locus of points at a given distance r from a given point C is the circle of radius r with the center at the point C. As it follows from the theorems of §56, §57:

The geometric locus of points equidistant from two given points is the perpendicular to the segment connecting these points, passing through the midpoint of the segment.

The geometric locus of interior points of an angle equidistant from its sides is the bisector of this angle.

59. The inverse theorem. If the hypothesis and the conclusion of a theorem are the *negations* of the hypothesis and the conclusion of another theorem, then the former theorem is called **inverse** to the latter one. For instance, the theorem inverse to: "if the digit sum is divisible by 9, then the number is divisible by 9" is: "if the digit sum is not divisible by 9, then the number is not divisible by 9."

It is worth mentioning that the validity of a direct theorem does not guarantee the validity of the inverse one: for example, the inverse proposition "if not every summand is divisible by a certain number then the sum is not divisible by this number" is false while the direct proposition is true.

The theorem described in §57 (both for the segment and for the angle) is inverse to the (direct) theorem described in §56.

60. Relationships between the theorems: direct, converse, inverse, and contrapositive. For better understanding of the relationship let us denote the hypothesis of the direct theorem by the letter A, and the conclusion by the letter B, and express the theorems concisely as:

(1) **Direct theorem**: if A is true, then B is true;
(2) **Converse theorem**: if B is true, then A is true;
(3) **Inverse theorem**: if A is false, then B is false;
(4) **Contrapositive theorem**: if B is false, then A is false.

Considering these propositions it is not hard to notice that the first one is in the same relationship to the fourth as the second one to the third. Namely, the propositions (1) and (4) can be transformed into each other, and so can the propositions (2) and (3). Indeed, from the proposition: "if A is true, then B is true" it follows immediately that "if B is false, then A is false" (since if A were true, then by the first proposition B would have been true too); and *vice versa*, from the proposition: "if B is false, then A is false" we derive: "if A is true, then B is true" (since if B were false, then A would have been false as well). Quite similarly, we can check that the second proposition follows from the third one, and *vice versa*.

Thus in order to make sure that all the four theorems are valid, there is no need to prove each of them separately, but it suffices to prove only two of them: direct and converse, or direct and inverse.

EXERCISES

104. Prove as a direct theorem that a point not lying on the perpendicular bisector of a segment is not equidistant from the endpoints of the segment; namely it is closer to that endpoint which lies on the same side of the bisector.

105. Prove as a direct theorem that any interior point of an angle which does not lie on the bisector is not equidistant from the sides of the angle.

106. Prove that two perpendiculars to the sides of an angle erected at equal distances from the vertex meet on the bisector.

107. Prove that if A and A', and B and B' are two pairs of points symmetric about some line XY, then the four points A, A', B', B lie on the same circle.

108. Find the geometric locus of vertices of isosceles triangles with a given base.

109. Find the geometric locus of the vertices A of triangles ABC with the given base BC and such that $\angle B > \angle C$.

110. Find the geometric locus of points equidistant from two given intersecting infinite straight lines.

111.* Find the geometric locus of points equidistant from three given infinite straight lines, intersecting pairwise.

112. For theorems from §60: direct, converse, inverse, and contrapositive, compare in which of the following four cases each of them is true: when (a) A is true and B is true, (b) A is true but B is false, (c) A is false but B is true, and (d) A is false and B is false.

113. By definition, the **negation** of a proposition is true whenever the proposition is false, and false whenever the proposition is true. State the negation of the proposition: "the digit sum of every multiple of 3 is divisible by 9." Is this proposition true? Is its negation true?

114. Formulate affirmatively the negations of the propositions: (a) in every quadrilateral, both diagonals lie inside it; (b) in every quadrilateral, there is a diagonal that lies inside it; (c) there is a quadrilateral whose both diagonals lie inside it; (d) there is a quadrilateral that has a diagonal lying outside it. Which of these propositions are true?

10 Basic construction problems

61. Preliminary remarks. Theorems we proved earlier allow us to solve some **construction** problems. Note that in elementary geometry one considers those constructions which can be performed using only *straightedge and compass.* [6]

62. Problem 1. ***To construct a triangle with the given three sides a, b and c*** (Figure 65).

[6] As we will see, the use of the drafting triangle, which can be allowed for saving time in the actual construction, is unnecessary in principle.

On any line MN, mark the segment CB congruent to one of the given sides, say, a. Describe two arcs centered at the points C and B of radii congruent to b and to c. Connect the point A, where these arcs intersect, with B and with C. The required triangle is ABC.

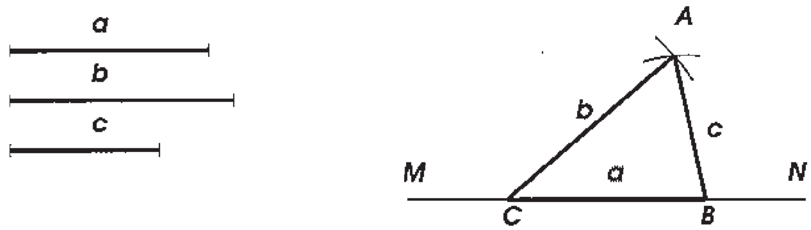

Figure 65

Remark. For three segments to serve as sides of a triangle, it is necessary that the greatest one is smaller than the sum of the other two (§48).

63. Problem 2. ***To construct an angle congruent to the given angle ABC and such that one of the sides is a given line MN, and the vertex is at a point O given on the line*** (Figure 66).

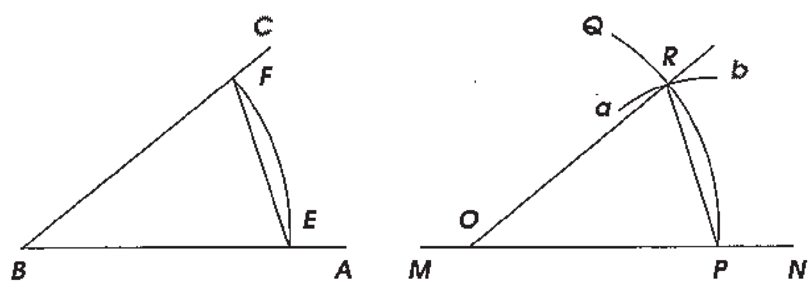

Figure 66

Between the sides of the given angle, describe an arc EF of any radius centered at the vertex B, then keeping the same setting of the compass place its pin leg at the point O and describe an arc PQ. Furthermore, describe an arc ab centered at the point P with the radius equal to the distance between the points E and F. Finally draw a line through O and the point R (the intersection of the two arcs). The angle ROP is congruent to the angle ABC because the triangles ROP and FBE are congruent as having congruent respective sides.

64. Problem 3. ***To bisect a given angle*** (Figure 67), or in other words, ***to construct the bisector of a given angle*** or ***to draw its axis of symmetry.***

Between the sides of the angle, draw an arc DE of arbitrary radius centered at the vertex B. Then, setting the compass to an arbitrary radius, greater however than half the distance between D and E (see Remark to Problem 1), describe two arcs centered at D and E so that they intersect at some point F. Drawing the line BF we obtain the bisector of the angle ABC.

For the proof, connect the point F with D and E by segments. We obtain two triangles BEF and BDF which are congruent since BF is their common side, and $BD = BE$ and $DE = EF$ by construction. The congruence of the triangles implies: $\angle ABF = \angle CBF$.

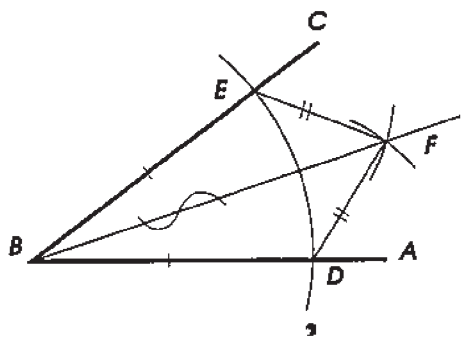

Figure 67

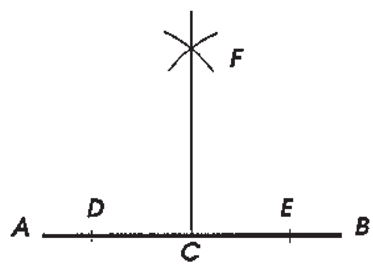

Figure 68

65. Problem 4. ***From a given point C on the line AB, to erect a perpendicular to this line*** (Figure 68).

On both sides of the point C on the line AB, mark congruent segments CD and CE (of any length). Describe two arcs centered at D and E of the same radius (greater than CD) so that the arcs intersect at a point F. The line passing through the points C and F will be the required perpendicular.

Indeed, as it is evident from the construction, the point F will have the same distance from the points D and E; therefore it will lie on the perpendicular to the segment AB passing through its midpoint (§56). Since the midpoint is C, and there is only one line passing through C and F, then $FC \perp DE$.

66. Problem 5. ***From a given point A, to drop a perpendicular to a given line BC*** (Figure 69).

Draw an arc of arbitrary radius (greater however than the distance from A to BC) with the center at A so that it intersects BC at some points D and E. With these points as centers, draw two arcs of the same arbitrary radius (greater however than $\frac{1}{2}DE$) so that they intersect at some point F. The line AF is the required perpendicular.

Indeed, as it is evident from the construction, each of the points A and F is equidistant from D and E, and all such points lie on the perpendicular to the segment AB passing through its midpoint (§58).

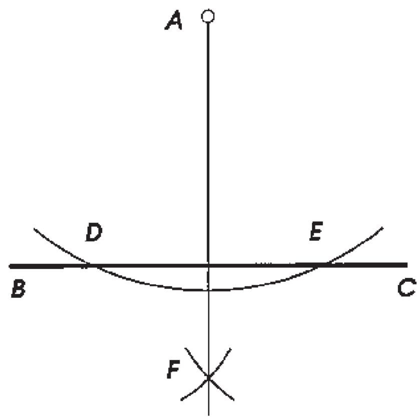

Figure 69

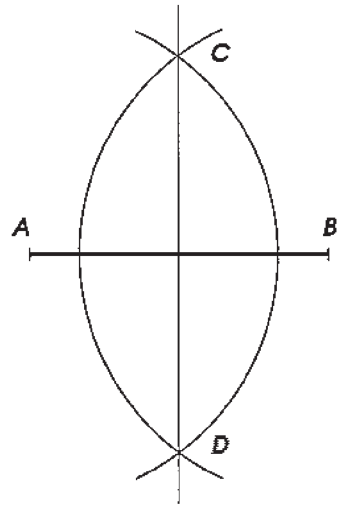

Figure 70

67. Problem 6. ***To draw the perpendicular to a given segment AB through its midpoint*** (Figure 70); in other words, ***to construct the axis of symmetry of the segment AB.***

Draw two arcs of the same arbitrary radius (greater than $\frac{1}{2}AB$), centered at A and B, so that they intersect each other at some points C and D. The line CD is the required perpendicular.

Indeed, as it is evident from the construction, each of the points C and D is equidistant from A and B, and therefore must lie on the symmetry axis of the segment AB.

Problem 7. ***To bisect a given straight segment*** (Figure 70). It is solved the same way as the previous problem.

68. Example of a more complex problem. The basic constructions allow one to solve more complicated construction problems. As an illustration, consider the following problem.

Problem. *To construct a triangle with a given base b, an angle α at the base, and the sum s of the other two sides* (Figure 71). To work out a solution plan, suppose that the problem has been solved, i.e. that a triangle ABC has been found such that the base $AC = b$, $\angle A = \alpha$ and $AB + BC = s$. Examine the obtained diagram. We know how to construct the side AC congruent to b and the angle A congruent to α. Therefore it remains on the other side of the angle to find a point B such that the sum $AB + BC$ is congruent to s.

Continuing AB past B, mark the segment AD congruent to s. Now the problem reduces to finding on AD a point B which would be the same distance away from C and D. As we know (§58), such a point must lie on the perpendicular to CD passing through its midpoint. The point will be found at the intersection of this perpendicular with AD.

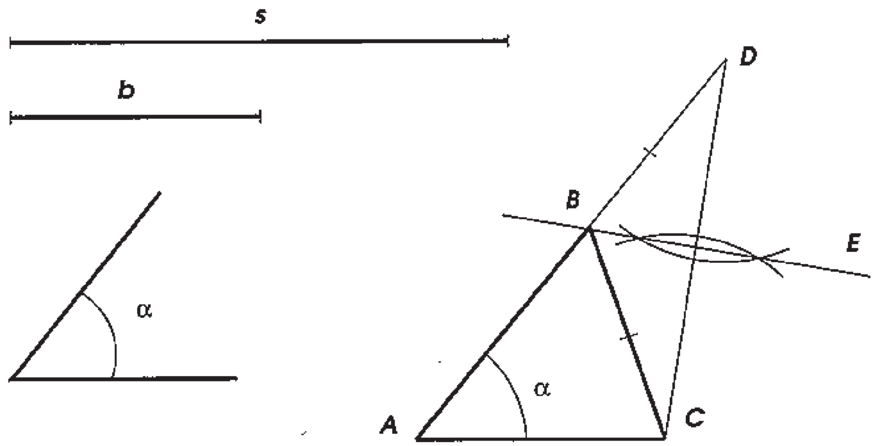

Figure 71

Thus, here is the solution of the problem: construct (Figure 71) the angle A congruent to α. On its sides, mark the segments $AC = b$ and $AD = s$, and connect the point D with C. Through the midpoint of CD, construct the perpendicular BE. Connect its intersection with AD, i.e. the point B, with C. The triangle ABC is a solution of the problem since $AC = b$, $\angle A = \alpha$ and $AB + BC = s$ (because $BD = BC$).

Examining the construction we notice that it is not always possible. Indeed, if the sum s is too small compared to b, then the perpendicular EB may miss the segment AD (or intersect the continuation of AD past A or past D). In this case the construction turns out *impossible*. Moreover, independently of the construction procedure, one can see that the problem has no solution if $s < b$ or $s = b$, because there is no triangle in which the sum of two sides is smaller than or congruent to the third side.

In the case when a solution exists, it turns out to be unique, i.e. there exists only one triangle, [7] satisfying the requirements of the

[7]There are infinitely many triangles satisfying the requirements of the problem, but they are all congruent to each other, and so it is customary to say that the solution of the problem is unique.

problem, since the perpendicular BE can intersect AD at one point at most.

69. Remark. The previous example shows that solution of a complex construction problem should consist of the following four stages.

(1) Assuming that the problem has been solved, we can draft the diagram of the required figure and, carefully examining it, try to find those relationships between the given and required data that would allow one to reduce the problem to other, previously solved problems. This most important stage, whose aim is to work out a plan of the solution, is called **analysis**.

(2) Once a plan has been found, the **construction** following it can be executed.

(3) Next, to validate the plan, one shows on the basis of known theorems that the constructed figure does satisfy the requirements of the problem. This stage is called **synthesis**.

(4) Then we ask ourselves: if the problem has a solution for any given data, if a solution is unique or there are several ones, are there any special cases when the construction simplifies or, on the contrary, requires additional examination. This solution stage is called **research**.

When a problem is very simple, and there is no doubt about possibility of the solution, then one usually omits the analysis and research stages, and provides only the construction and the proof. This was what we did describing our solutions of the first seven problems of this section; this is what we are going to do later on whenever the problems at hand will not be too complex.

EXERCISES

Construct:

115. The sum of two, three, or more given angles.

116. The difference of two angles.

117. Two angles whose sum and difference are given.

118. Divide an angle into 4, 8, 16 congruent parts.

119. A line in the exterior of a given angle passing through its vertex and such that it would form congruent angles with the sides of this angle.

120. A triangle: (a) given two sides and the angle between them; (b) given one side and both angles adjacent to it; (c) given two sides and the angle opposite to the greater one of them; (d) given two sides and the angle opposite to the smaller one of them (in this case there can be two solutions, or one, or none).

121. An isosceles triangle: (a) given its base and another side; (b) given its base and a base angle; (c) given its base angle and the opposite side.

122. A right triangle: (a) given both of its legs; (b) given one of the legs and the hypotenuse; (c) given one of the legs and the adjacent acute angle.

123. An isosceles triangle: (a) given the altitude to the base and one of the congruent sides; (b) given the altitude to the base and the angle at the vertex; (c) given the base and the altitude to another side.

124. A right triangle, given an acute angle and the hypotenuse.

125. Through an interior point of an angle, construct a line that cuts off congruent segments on the sides of the angle.

126. Through an exterior point of an angle, construct a line which would cut off congruent segments on the sides of the angle.

127. Find two segments whose sum and difference are given.

128. Divide a given segment into 4, 8, 16 congruent parts.

129. On a given line, find a point equidistant from two given points (outside the line).

130. Find a point equidistant from the three vertices of a given triangle.

131. On a given line intersecting the sides of a given angle, find a point equidistant from the sides of the angle.

132. Find a point equidistant from the three sides of a given triangle.

133. On an infinite line AB, find a point C such that the rays CM and CN connecting C with two given points M and N situated on the same side of AB would form congruent angles with the rays CA and CB respectively.

134. Construct a right triangle, given one of its legs and the sum of the other leg with the hypotenuse.

135. Construct a triangle, given its base, one of the angles adjacent to the base, and the difference of the other two sides (consider two cases: (1) when the smaller of the two angles adjacent to the base is given; (2) when the greater one is given).

136. Construct a right triangle, given one of its legs and the difference of the other two sides.

137. Given an angle A and two points B and C situated one on one side of the angle and one on the other, find: (1) a point M equidistant from the sides of the angle and such that $MB = MC$; (2) a point N equidistant from the sides of the angle and such that $NB = BC$; (3) a point P such that each of the points B and C would be the same distance away from A and P.

138. Two towns are situated near a straight railroad line. Find the position for a railroad station so that it is equidistant from the towns.

139. Given a point A on one of the sides of an angle B. On the other side of the angle, find a point C such that the sum $CA + CB$ is congruent to a given segment.

11 Parallel lines

70. `Definitions.` Two lines are called **parallel** if they lie in the same plane and do not intersect one another no matter how far they are extended in both directions.

In writing, parallel lined are denoted by the symbol $\|$. Thus, if two lines AB and CD are parallel, one writes $AB\|CD$.

Existence of parallel lines is established by the following theorem.

71. `Theorem.` ***Two perpendiculars*** (AB and CD, Figure 72) ***to the same line*** (MN) ***cannot intersect no matter how far they are extended.***

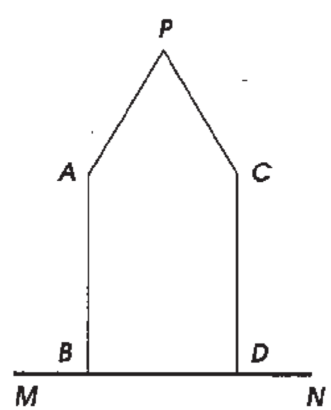

Figure 72

Indeed, if such perpendiculars could intersect at some point P, then two perpendiculars to the line MN would be dropped from this point, which is impossible (§24). Thus two perpendiculars to the same line are parallel to each other.

72. Names of angles formed by intersection of two lines by a transversal. Let two lines AB and CD (Figure 73) be intersected by a third line MN. Then 8 angles are formed (we labeled them by numerals) which carry pairwise the following names:

corresponding angles: 1 and 5, 4 and 8, 2 and 6, 3 and 7;

alternate angles: 3 and 5, 4 and 6 (interior); 1 and 7, 2 and 8 (exterior);

same-side angles: 4 and 5, 3 and 6 (interior); 1 and 8, 2 and 7 (exterior).

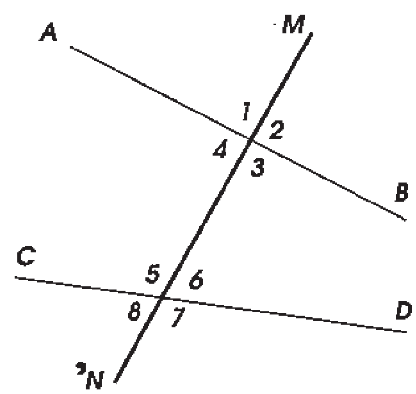

Figure 73

73. Tests for parallel lines. ***When two lines*** (AB and CD, Figure 74) ***are intersected by a third line*** (MN), ***and it turns out that:***

(1) ***some corresponding angles are congruent,*** or

(2) ***some alternate angles are congruent,*** or

(3) ***the sum of some same-side interior or same-side exterior angles is*** $2d$,

then these two lines are parallel.

Suppose, for example, that the corresponding angles 2 and 6 are congruent. We are required to show that in this case $AB \| CD$. Let us assume the contrary, i.e. that the lines AB and CD are not parallel. Then these lines intersect at some point P lying on the right of MN or at some point P' lying on the left of MN. If the intersection is at P, then a triangle is formed for which the angle 2 is exterior, and the angle 6 interior not supplementary to it. Therefore the angle 2 has to be greater than the angle 6 (§42), which contradicts the hypothesis. Thus the lines AB and CD cannot intersect at any point P on the right of MN. If we assume that the intersection is at the point P', then a triangle is formed for which the angle 4, congruent to the angle 2, is interior and the angle 6 is exterior not supplementary to it. Then the angle 6 has to be greater than the angle 4, and hence greater than the angle 2, which contradicts the hypothesis. Therefore the lines AB and CD cannot intersect at a point lying on the left of MN either. Thus the lines cannot intersect anywhere, i.e. they are parallel. Similarly, one can prove that $AB \| CD$ if $\angle 1 = \angle 5$, or $\angle 3 = \angle 7$, etc.

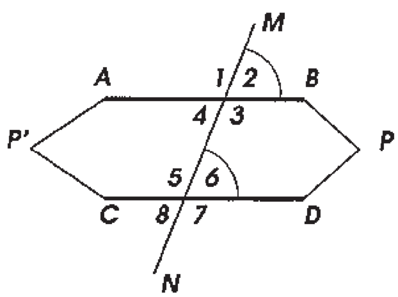

Figure 74

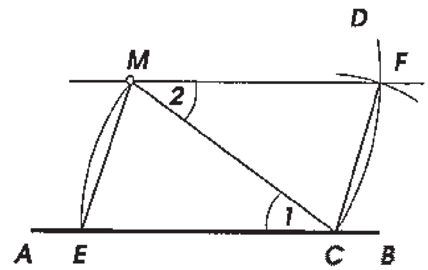

Figure 75

Suppose now that $\angle 4 + \angle 5 = 2d$. Then we conclude that $\angle 4 = \angle 6$ since the sum of angle 6 with the angle 5 is also $2d$. But if $\angle 4 = \angle 6$, then the lines AB and CD cannot intersect, since if they did the angles 4 and 6 (of which one would have been exterior and the other interior not supplementary to it) could not be congruent.

74. Problem. *Through a given point M* (Figure 75), *to construct a line parallel to a given line AB.*

A simple solution to this problem consists of the following. Draw an arc CD of arbitrary radius centered at the point M. Next, draw the arc ME of the same radius centered at the point C. Then draw a small arc of the radius congruent to ME centered at the point C so that it intersects the arc CD at some point F. The line MF will be parallel to AB.

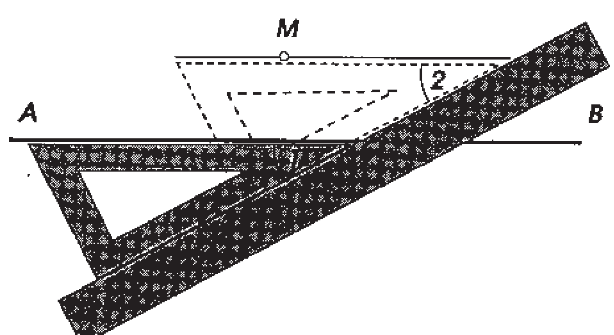

Figure 76

To prove this, draw the auxiliary line MC. The angles 1 and 2 thus formed are congruent by construction (because the triangles EMC and MCF are congruent by the SSS-test), and when alternate angles are congruent, the lines are parallel.

For practical construction of parallel lines it is also convenient to use a drafting triangle and a straightedge as shown in Figure 76.

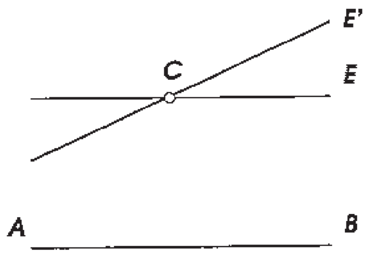

Figure 77

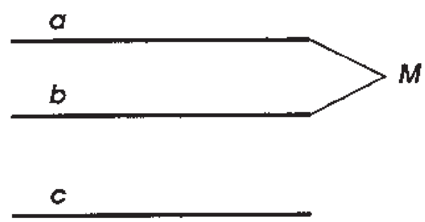

Figure 78

75. The parallel postulate. ***Through a given point, one cannot draw two different lines parallel to the same line.***

Thus, if (Figure 77) $CE \| AB$, then no other line CE' passing through the point C can be parallel to AB, i.e. CE' will meet AB when extended.

It turns out impossible to prove this proposition, i.e. to derive it as a consequence of earlier accepted axioms. It becomes necessary therefore to accept it as a new assumption (postulate, or axiom).

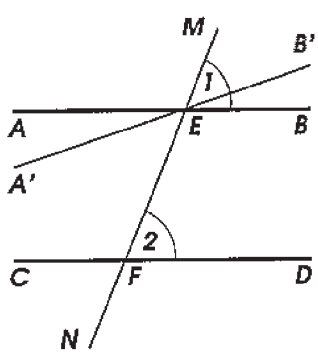

Figure 79

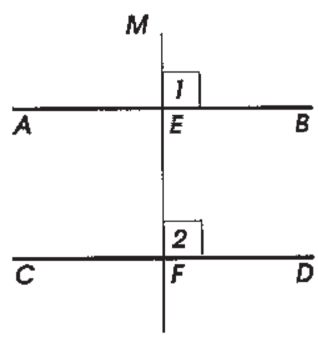

Figure 80

76. Corollary. (1) *If* $CE \| AB$ (Figure 77), *and a third line* CE' *intersects one of these two parallel lines, then it intersects the other as well,* because otherwise there would be two different lines CE and CE' passing through the same point C and parallel to AB, which is impossible.

(2) *If each of two lines a and b* (Figure 78) *is parallel to the same third line c, then they are parallel to each other.*

Indeed, if we assume that the lines a and b intersect at some point M, there would be two different lines passing through this point and parallel to c, which is impossible.

77. Angles formed by intersection of parallel lines by a transversal.

Theorem (converse to Theorem of §73). ***If two parallel lines*** (AB and CD, Figure 79) ***are intersected by any line*** (MN), ***then:***

(1) ***corresponding angles are congruent;***

(2) ***alternate angles are congruent;***

(3) ***the sum of same-side interior angles is*** $2d$;

(4) ***the sum of same-side exterior angles is*** $2d$.

Let us prove for example that if $AB \| CD$, then the corresponding angles a and b are congruent.

Assume the contrary, i.e. that these angles are not congruent (let us say $\angle 1 > \angle 2$). Constructing $\angle MEB' = \angle 2$ we then obtain a line $A'B'$ distinct from AB and have therefore two lines passing through the point E and parallel to the same line CD. Namely, $AB \| CD$ by the hypothesis of the theorem, and $A'B' \| CD$ due to the congruence of the corresponding angles MEB' and 2. Since this contradicts the parallel postulate, then our assumption that the angles 1 and 2 are not congruent must be rejected; we are left to accept that $\angle 1 = \angle 2$.

Other conclusions of the theorem can be proved the same way.

Corollary. *A perpendicular to one of two parallel lines is perpendicular to the other one as well.*

Indeed, if $AB \| CD$ (Figure 80) and $ME \perp AB$, then firstly ME, which intersects AB, will also intersect CD at some point F, and secondly the corresponding angles 1 and 2 will be congruent. But the angle 1 is right, and thus the angle 2 is also right, i.e. $ME \perp CD$.

78. Tests for non-parallel lines. From the two theorems: direct (§73) and its converse (§75), it follows that the inverse theorems also hold true, i.e.:

If two lines are intersected by a third one in a way such that (1) *corresponding angles are not congruent*, or (2) *alternate interior angles are not congruent*, etc., *then the two lines are not parallel;*

If two lines are not parallel and are intersected by a third one, then (1) *corresponding angles are not congruent,* (2) *alternate interior angles are not congruent,* etc. Among all these tests for non-parallel lines (which are easily proved by *reductio ad absurdum*), the following one deserves special attention:

If the sum of two same-side interior angles (1 and 2, Figure 81) ***differs from*** $2d$***, then the two lines when extended far enough will intersect,*** since if these lines did not intersect, then they would be parallel, and then the sum of same-side interior angles would be $2d$, which contradicts the hypothesis.

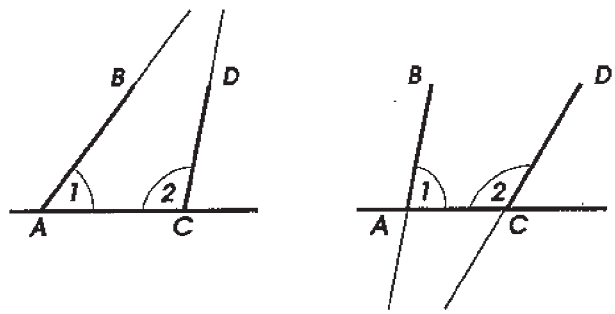

Figure 81

This proposition (supplemented by the statement that the lines intersect on that side of the transversal on which the sum of the same-side interior angles is *smaller than* $2d$) was accepted without proof by the famous Greek geometer **Euclid** (who lived in the 3rd century B.C.) in his ***Elements*** of geometry, and is known as **Euclid's postulate.** Later the preference was given to a simpler formulation: the parallel postulate stated in §75.

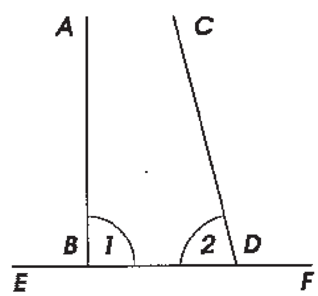

Figure 82

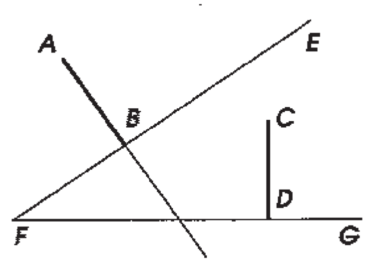

Figure 83

Let us point out two more tests for non-parallelism which will be used later on:

(1) ***A perpendicular*** (AB, Figure 82) ***and a slant*** (CD) ***to the same line*** (EF) ***intersect each other,*** because the sum of same-side interior angles 1 and 2 differs from $2d$.

(2) ***Two lines*** (AB and CD, Figure 83) ***perpendicular to two intersecting lines*** (FE and FG) ***intersect as well.***

Indeed, if we assume the contrary, i.e. that $AB \| CD$, then the line FD, being perpendicular to one of the parallel lines (CD), will be perpendicular to the other (AB), and thus two perpendiculars from the same point F to the same line AB will be dropped, which is impossible.

79. Angles with respectively parallel sides.

Theorem. ***If the sides of one angle are respectively parallel to the sides of another angle, then such angles are either congruent or add up to*** $2d$.

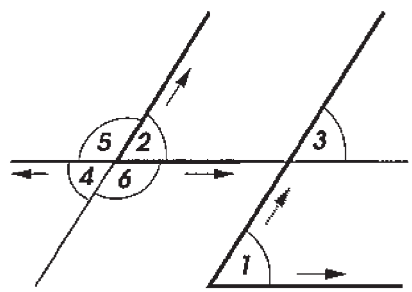

Figure 84

Consider separately the following three cases (Figure 84).

(1) Let the sides of the angle 1 be respectively parallel to the sides of the angle 2 and, beside this, the directions of the respective sides, when counted away from the vertices (as indicated by arrows on the diagram), happen to be the same.

Extending one of the sides of the angle 2 until it meets the non-parallel to it side of the angle 1, we obtain the angle 3 congruent to each of the angles 1 and 2 (as corresponding angles formed by a transversal intersecting parallel lines). Therefore $\angle 1 = \angle 2$.

(2) Let the sides of the angle 1 be respectively parallel to the sides of the angle 2, but the respective sides have opposite directions away from the vertices.

Extending both sides of the angle 4, we obtain the angle 2, which is congruent to the angle 1 (as proved earlier) and to the angle 4 (as vertical to it). Therefore $\angle 4 = \angle 1$.

(3) Finally, let the sides of the angle 1 be respectively parallel to the sides of the angles 5 and 6, and one pair of respective sides have

the same directions, while the other pair, the opposite ones.

Extending one side of the angle 5 or the angle 6, we obtain the angle 2, congruent (as proved earlier) to the angle 1. But $\angle 5$(or $\angle 6$) + $\angle 2 = 2d$ (by the property of supplementary angles). Therefore $\angle 5$(or $\angle 6$) + $\angle 1 = 2d$ too.

Thus angles with parallel sides turn out to be congruent when the directions of respective sides away from the vertices are either both the same or both opposite, and when neither condition is satisfied, the angles add up to $2d$.

Remark. One could say that two angles with respectively parallel sides are congruent when both are acute or both are obtuse. In some cases however it is hard to determine *a priori* if the angles are acute or obtuse, so comparing directions of their sides becomes necessary.

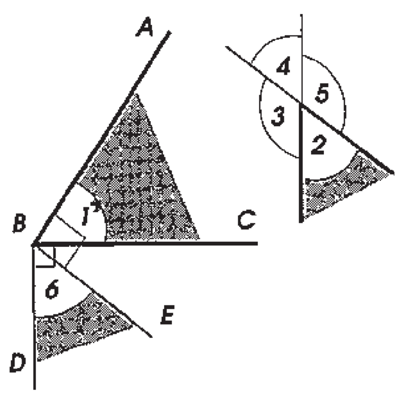

Figure 85

80. Angles with respectively perpendicular sides.

Theorem. ***If the sides of one angle are respectively perpendicular to the sides of another one, then such angles are either congruent or add up to*** $2d$.

Let the angle ABC labeled by the number 1 (Figure 85) be one of the given angles, and the other be one of the four angles 2, 3, 4, 5 formed by two intersecting lines, of which one is perpendicular to the side AB and the other to the side BC.

From the vertex of the angle 1, draw two auxiliary lines: $BD \perp BC$ and $BE \perp BA$. The angle 6 formed by these lines is congruent to the angle 1 for the following reason. The angles DBC and EBA are congruent since both are right. Subtracting from each of them the same angle EBC we obtain: $\angle 1 = \angle 6$. Now notice that the sides of the auxiliary angle 6 are parallel to the intersecting lines which form the angles 2, 3, 4, 5 (because two perpendiculars to the same line are parallel, §71). Therefore the latter angles are either congruent to the angle 6 or supplement it to $2d$. Replacing the angle 6 with the angle 1 congruent to it, we obtain what was required to prove.

EXERCISES

140. Divide the plane by infinite straight lines into five parts, using as few lines as possible.

141. In the interior of a given angle, construct an angle congruent to it.

142. Using a protractor, straightedge, and drafting triangle, measure an angle whose vertex does not fit the page of the diagram.

143. How many axes of symmetry does a pair of parallel lines have? How about three parallel lines?

144. Two parallel lines are intersected by a transversal, and one of the eight angles thus formed is 72°. Find the measures of the remaining seven angles.

145. One of the interior angles formed by a transversal with one of two given parallel lines is $4d/5$. What angle does its bisector make with the other of the two parallel lines?

146. The angle a transversal makes with one of two parallel lines is by 90° greater than with the other. Find the angle.

147. Four out of eight angles formed by a transversal intersecting two given lines contain 60° each, and the remaining four contain 120° each. Does this imply that the given lines are parallel?

148. At the endpoints of the base of a triangle, perpendiculars to the lateral sides are erected. Compute the angle at the vertex of the triangle if these perpendiculars intersect at the angle of 120°.

149. Through a given point, construct a line making a given angle to a given line.

150. Prove that if the bisector of one of the exterior angles of a triangle is parallel to the opposite side, then the triangle is isosceles.

151. In a triangle, through the intersection point of the bisectors of the angles adjacent to a base, a line parallel to the base is drawn. Prove that the segment of this line contained between the lateral sides of the triangle is congruent to the sum of the segments cut out on these sides and adjacent to the base.

*152.** Bisect an angle whose vertex does not fit the page of the diagram.

12 The angle sum of a polygon

81. Theorem. ***The sum of angles of a triangle is** 2d.*

Let ABC (Figure 86) be any triangle; we are required to prove that the sum of the angles A, B and C is $2d$, i.e. 180°.

Extending the side AC past C and drawing $CE \| AB$ we find: $\angle A = \angle ECD$ (as corresponding angles formed by a transversal intersecting parallel lines) and $\angle B = \angle BCE$ (as alternate angles formed by a transversal intersecting parallel lines). Therefore

$$\angle A + \angle B + \angle C = \angle ECD + \angle BCE + \angle C = 2d = 180^\circ.$$

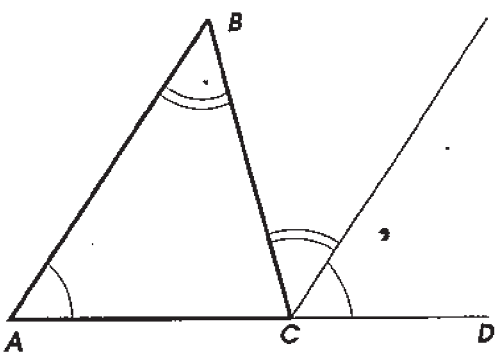

Figure 86

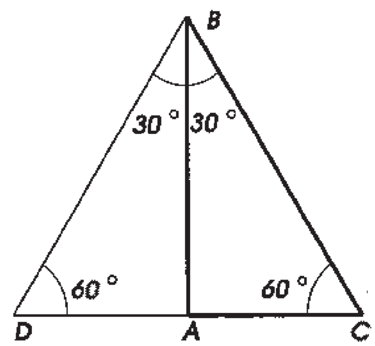

Figure 87

Corollaries. (1) *Any exterior angle of a triangle is congruent to the sum of the interior angles not supplementary to it* (e.g. $\angle BCD = \angle A + \angle B$).

(2) *If two angles of one triangle are congruent respectively to two angles of another, then the remaining angles are congruent as well.*

(3) *The sum of the two acute angles of a right triangle is congruent to one right angle, i.e. it is* 90°.

(4) *In an isosceles right triangle, each acute angle is* $\frac{1}{2}d$, *i.e.* 45°.

(5) *In an equilateral triangle, each angle is* $\frac{2}{3}d$, *i.e.* 60°.

(6) *If in a right triangle* ABC (Figure 87) *one of the acute angles* (for instance, $\angle B$) *is* 30°, *then the leg opposite to it is congruent to a half of the hypotenuse.* Indeed, noticing that the other acute angle in such a triangle is 60°, attach to the triangle ABC another triangle ABD congruent to it. Then we obtain the triangle DBC, whose angles are 60° each. Such a triangle has to be equilateral (§45), and hence $DC = BC$. But $AC = \frac{1}{2}DC$, and therefore $AC = \frac{1}{2}BC$.

We leave it to the reader to prove the converse proposition: *If a leg is congruent to a half of the hypotenuse, then the acute angle opposite to it is* 30°.

82. Theorem. ***The sum of angles of a convex polygon having n sides is congruent to two right angles repeated $n-2$ times.***

Taking, inside the polygon, an arbitrary point O (Figure 88), connect it with all the vertices. The convex polygon is thus partitioned into as many triangles as it has sides, i.e. n. The sum of angles in each of them is $2d$. Therefore the sum of angles of all the triangles is $2dn$. Obviously, this quantity exceeds the sum of all angles of the polygon by the sum of all those angles which are situated around the point O. But the latter sum is $4d$ (§27). Therefore the sum of angles of the polygon is

$$2dn - 4d = 2d(n-2) = 180^\circ \times (n-2).$$

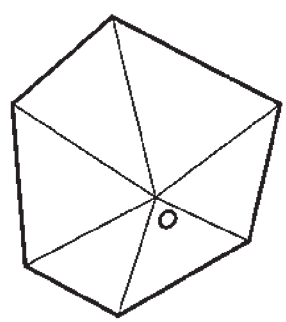

Figure 88

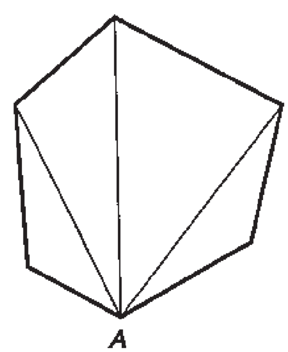

Figure 89

Remarks. (1) The theorem can be also proved this way. From any vertex A (Figure 89) of the convex polygon, draw its diagonals. The polygon is thus partitioned into triangles, the number of which is two less than the number of sides of the polygon. Indeed, if we exclude from counting those two sides which form the angle A of the polygon, then the remaining sides correspond to one triangle each. Therefore the total number of such triangles is $n-2$, where n denotes the number of sides of the polygon. In each triangle, the sum of angles is $2d$, and hence the sum of angles of all the triangles is $2d(n-2)$. But the latter sum is the sum of all angles of the polygon.

(2) The same result holds true for any non-convex polygon. To prove this, one should first partition it into convex ones. For this, it suffices to extend all sides of the polygon in both directions. The

infinite straight lines thus obtained will divide the plane into convex parts: convex polygons and some infinite regions. The original non-convex polygon will consist of some of these convex parts.

83. Theorem. ***If at each vertex of a convex polygon, we extend one of the sides of this angle, then the sum of the exterior angles thus formed is congruent to*** $4d$ (regardless of the number of sides of the polygon).

Each of such exterior angles (Figure 90) supplements to 2d one of the interior angles of the polygon. Therefore if to the sum of all interior angles we add the sum of these exterior angles, the result will be $2dn$ (where n is the number of sides of the polygon). But the sum of the interior angles, as we have seen, is $2dn - 4d$. Therefore the sum of the exterior angles is the difference:

$$2dn - (2db - 4d) = 2dn - 2dn + 4d = 4d = 360°.$$

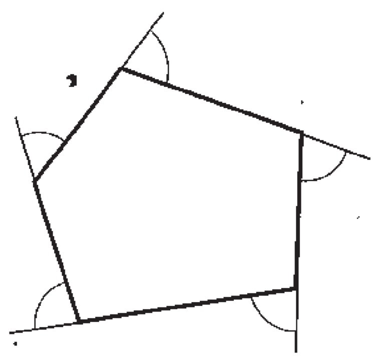

Figure 90

EXERCISES

153. Compute the angle between two medians of an equilateral triangle.

154. Compute the angle between bisectors of acute angles in a right triangle.

155. Given an angle of an isosceles triangle, compute the other two. Consider two cases: the given angle is (a) at the vertex, or (b) at the base.

156. Compute interior and exterior angles of an equiangular pentagon.

157.* Compute angles of a triangle which is divided by one of its bisectors into two isosceles triangles. Find all solutions.

158. Prove that if two angles and the side opposite to the first of them in one triangle are congruent respectively to two angles and the side opposite to the first of them in another triangle, then such triangles are congruent.
`Remark:` This proposition is called sometimes the **AAS-test**, or **SAA-test**.

159. Prove that if a leg and the acute angle opposite to it in one right triangle are congruent respectively to a leg and the acute angle opposite to it in another right triangle, then such triangles are congruent.

160. Prove that in a convex polygon, one of the angles between the bisectors of two consecutive angles is congruent to the semisum of these two angles.

161. Given two angles of a triangle, construct the third one.

162. Given an acute angle of a right triangle, construct the other acute angle.

163. Construct a right triangle, given one of its legs and the acute angle opposite to it.

164. Construct a triangle, given two of its angles and a side opposite to one of them.

165. Construct an isosceles triangle, given its base and the angle at the vertex.

166. Construct an isosceles triangle: (a) given the angle at the base, and the altitude dropped to one of the lateral sides; (b) given the lateral side and the altitude dropped to it.

167. Construct an equilateral triangle, given its altitude.

168. Trisect a right angle (in other words, construct the angle of $\frac{1}{3} \times 90° = 30°$).

169. Construct a polygon congruent to a given one.
`Hint:` Diagonals partition a convex polygon into triangles.

170. Construct a quadrilateral, given three of its angles and the sides containing the fourth angle.
`Hint:` Find the fourth angle.

171.* How many acute angles can a convex polygon have?

172.* Find the sum of the "interior" angles at the five vertices of a five-point star (e.g. the one shown in Figure 221), and the sum of its five exterior angles (formed by extending one of the sides at each vertex). Compare the results with those of §82 and §83.

173.* Following Remark (2) in §82, extend the results of §82 and §83 to non-convex polygons.

13 Parallelograms and trapezoids

84. The parallelogram. A quadrilateral whose opposite sides are pairwise parallel is called a **parallelogram**. Such a quadrilateral ($ABCD$, Figure 91) is obtained, for instance, by intersecting any two parallel lines KL and MN with two other parallel lines RS and PQ.

85. Properties of sides and angles.

Theorem. ***In any parallelogram, opposite sides are congruent, opposite angles are congruent, and the sum of angles adjacent to one side is*** $2d$ (Figure 92).

Drawing the diagonal BD we obtain two triangles: ABD and BCD, which are congruent by the ASA-test because BD is their common side, $\angle 1 = \angle 4$, and $\angle 2 = \angle 3$ (as alternate angles formed by a transversal intersecting parallel lines). It follows from the congruence of the triangles that $AB = CD$, $AD = BC$, and $\angle A = \angle C$. The opposite angles B and D are also congruent since they are sums of congruent angles.

Finally, the angles adjacent to one side, e.g. the angles A and D, add up to $2d$ since they are same-side interior angles formed by a transversal intersecting parallel lines.

Corollary. *If one of the angles of a parallelogram is right, then the other three are also right.*

Remark. The congruence of the opposite sides of a parallelogram can be rephrased this way: *parallel segments cut out by parallel lines are congruent.*

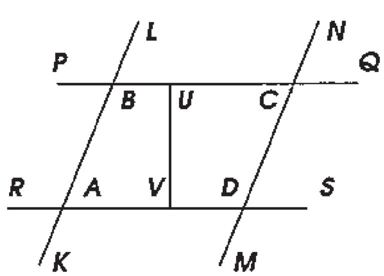

Figure 91

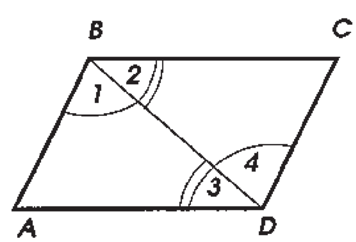

Figure 92

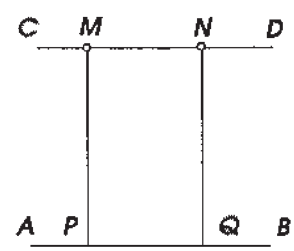

Figure 93

Corollary. *If two lines are parallel, then all points of each of them are the same distance away from the other line;* in short *parallel lines* (AB and CD, Figure 93) *are everywhere the same distance apart.*

Indeed, if from any two points M and N of the line CD, the perpendiculars MP and NQ to AB are dropped, then these perpendiculars are parallel (§71), and therefore the quadrilateral $MNQP$ is a parallelogram. It follows that $MN = NQ$, i.e. the points M and N are are the same distance away from the line AB.

Remark. Given a parallelogram ($ABCD$, Figure 91), one sometimes refers to a pair of its parallel sides (e.g. AD and BC) as a pair of **bases**. In this case, a line segment (UV) connecting the parallel lines PQ and RS and perpendicular to them is called an **altitude** of the parallelogram. Thus, the corollary can be rephrased this way: *all altitudes between the same bases of a parallelogram are congruent to each other.*

86. Two tests for parallelograms.

Theorem. ***If in a convex quadrilateral:***

(1) ***opposite sides are congruent to each other,*** or

(2) ***two opposite sides are congruent and parallel,***

then this quadrilateral is a parallelogram.

(1) Let $ABCD$ (Figure 92) be a quadrilateral such that

$$AB = CD \text{ and } BC = AD.$$

It is required to prove that this quadrilateral is a parallelogram, i.e. that $AB \| CD$ and $BC \| AD$.

Drawing the diagonal BD we obtain two triangles, which are congruent by the SSS-test since BD is their common side, and $AB = CD$ and $BC = AD$ by hypothesis. It follows from the congruence of the triangles that $\angle 1 = \angle 4$ and $\angle 2 = \angle 3$ (in congruent triangles, congruent sides oppose congruent angles). This implies that $AB \| CD$ and $BC \| AD$ (if alternate angles are congruent, then the lines are parallel).

(2) Let $ABCD$ (Figure 92) be a quadrilateral such that $BC \| AD$ and $BC = AD$. It is required to prove that $ABCD$ is a parallelogram, i.e. that $AB \| CD$.

The triangles ABD and BCD are congruent by the SAS-test because BD is their common side, $BC = AD$ (by hypothesis), and $\angle 2 = \angle 3$ (as alternate angles formed by intersecting parallel lines by a transversal). The congruence of the triangles implies that $\angle 1 = \angle 4$, and therefore $AB \| CD$.

87. The diagonals and their property.

Theorem. (1) ***If a quadrilateral*** ($ABCD$, Figure 94) ***is a parallelogram, then its diagonals bisect each other.***

(2) *Vice versa,* ***in a quadrilateral, if the diagonals bisect each other, then this quadrilateral is a parallelogram.***

(1) The triangles BOC and AOD are congruent by the ASA-test, because $BC = AD$ (as opposite sides of a parallelogram), $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (as alternate angles). It follows from the congruence of the triangles that $OA = OC$ and $OD = OB$.

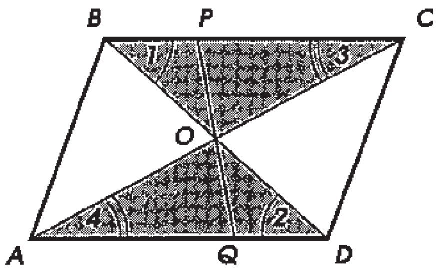

Figure 94

(2) If $AO = OC$ and $BO = OD$, then the triangles AOD and BOC are congruent (by the SAS-test). It follows from the congruence of the triangles that $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Therefore $BC \| AD$ (alternate angles are congruent) and $BC = AD$. Thus $ABCD$ is a parallelogram (by the second test).

88. Central symmetry. Two points A and A' (Figure 95) are called **symmetric** about a point O, if O is the midpoint of the line segment AA'.

Thus, in order to construct the point symmetric to a given point A about another given point O, one should connect the points A and O by a line, extend this line past the point O, and mark on the extension the segment OA' congruent to OA. Then A' is the required point.

Two figures (or two parts of the same figure) are called symmetric about a given point O, if for each point of one figure, the point symmetric to it about the point O belongs to the other figure, and *vice versa*. The point O is then called the **center of symmetry**. The symmetry itself is called *central* (as opposed to the *axial* symmetry we encountered in §37). If each point of a figure is symmetric to some point of the same figure (about a certain center), then the figure is said to have a center of symmetry. An example of such a figure is a circle; its center of symmetry is the center of the circle.

Every figure can be superimposed on the figure symmetric to it by rotating the figure through the angle 180° ***about the center of symmetry.*** Indeed, any two symmetric points (say, A and A', Figure 95) exchange their positions under this rotation.

Remarks. (1) Two figures symmetric about a point can be superimposed therefore by a motion *within* the plane, i.e. *without* lifting them off the plane. In this regard central symmetry differs from axial symmetry (§37), where for superimposing the figures it was necessary to flip one of them over.

(2) Just like axial symmetry, central symmetry is frequently found around us (see Figure 96, which indicates that each of the letters N and S has a center of symmetry while E and W do not).

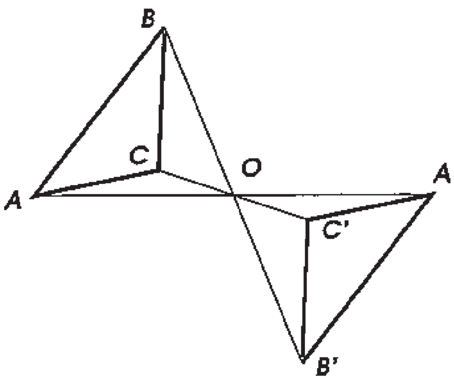

Figure 95

Figure 96

89. ***In a parallelogram, the intersection point of the diagonals is the center of symmetry*** (Figure 94).

Indeed, the vertices A and C are symmetric about the intersection point O of the diagonals (since $AO = OD$), and so are B and C. Furthermore, for a point P on the boundary of the parallelogram, draw the line PO, and let Q be the point where the extension of line past O meets the boundary. The triangles AQO and CPO are congruent by the ASA-test for $\angle 4 = \angle 3$ (as alternate), $\angle QOA = \angle POC$ (as vertical), and $AO = OC$. Therefore $QO = OP$, i.e. the points P and Q are symmetric about the center O.

Remark. If a parallelogram is turned around 180° about the intersection point of the diagonals, then each vertex exchanges its position with the opposite one (A with C, and B with D in Figure 94), and the new position of the parallelogram will coincide with the old one.

Most parallelograms do not possess axial symmetry. In the next section we will find out which of them do.

90. The rectangle and its properties. If one of the angles of a parallelogram is right then the other three are also right (§85). A parallelogram all of whose angles are right is called a **rectangle**.

Since rectangles are parallelograms, they possess all properties of parallelograms (for instance, their diagonals bisect each other, and the intersection point of the diagonals is the center of symmetry). However rectangles have their own special properties.

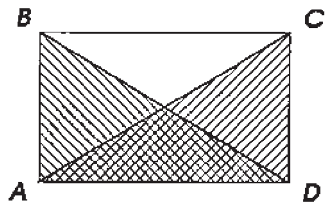

Figure 97

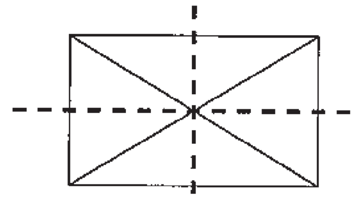

Figure 98

(1) ***In a rectangle*** ($ABCD$, Figure 97), ***the diagonals are congruent.***

The right triangles ACD and ABD are congruent because they have respectively congruent legs (AD is a common leg, and $AB = CD$ as opposite sides of a parallelogram). The congruence of the triangles implies: $AC = BD$.

(2) ***A rectangle has two axes of symmetry.*** Namely, each line passing through the center of symmetry and parallel to two opposite sides of the rectangle is its axis of symmetry. The axes of symmetry of a rectangle are perpendicular to each other (Figure 98).

91. The rhombus and its properties. A parallelogram all of whose sides are congruent is called a **rhombus**. Beside all the properties that parallelograms have, rhombi also have the following special ones.

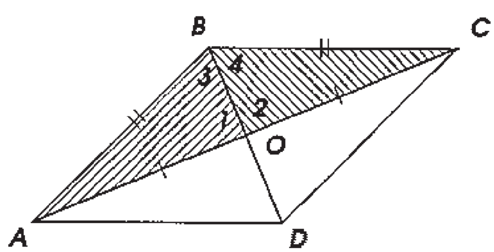

Figure 99

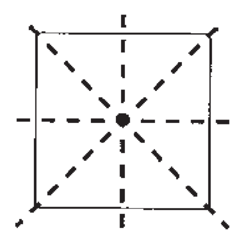

Figure 100

(1) ***Diagonals of a rhombus*** ($ABCD$, Figure 99) ***are perpendicular and bisect the angles of the rhombus.***

The triangles AOB and COB are congruent by the SSS-test because BO is their common side, $AB = BC$ (since all sides of a rhombus are congruent), and $AO = OC$ (since the diagonals of any parallelogram bisect each other). The congruence of the triangles implies that

$$\angle 1 = \angle 2, \quad \text{i.e.} \quad BD \perp AC, \quad \text{and} \quad \angle 3 = \angle 4,$$

i.e. the angle B is bisected by the diagonal BD. From the congruence of the triangles BOC and DOC, we conclude that the angle C is bisected by the diagonal CA, etc.

(2) ***Each diagonal of a rhombus is its axis of symmetry.***

The diagonal BD (Figure 99) is an axis of symmetry of the rhombus $ABCD$ because by rotating $\triangle BAD$ about BD we can superimpose it onto $\triangle BCD$. Indeed, the diagonal BD bisects the angles B and D, and beside this $AB = BC$ and $AD = DC$.

The same reasoning applies to the diagonal AC.

92. The square and its properties. **A square** can be defined as a parallelogram all of whose sides are congruent and all of whose angles are right. One can also say that a square is a rectangle all of whose sides are congruent, or a rhombus all of whose angles are right. Therefore a square possesses all the properties of parallelograms, rectangles and rhombi. For instance, a square has four axes of symmetry (Figure 100): two passing through the midpoints of opposite sides (as in a rectangle), and two passing through the vertices of the opposite angles (as in a rhombus).

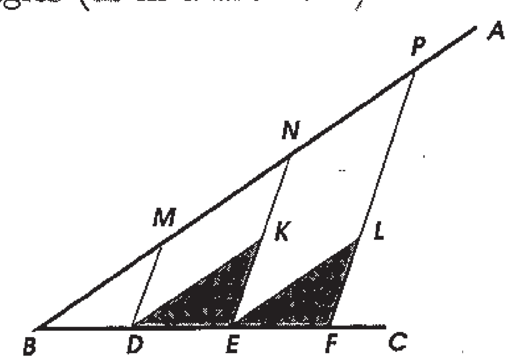

Figure 101

93. A theorem based on properties of parallelograms.

Theorem. *If on one side of an angle* (e.g. on the side BC of the angle ABC, Figure 101), *we mark segments congruent to each other* ($DE = EF = \ldots$), *and through their endpoints, we draw parallel lines* (DM, EN, FP, $\ldots$) *until their intersections with the other side of the angle, then the segments cut out on this side will be congruent to each other* ($MN = NP = \ldots$).

Draw the auxiliary lines DK and DL parallel to AB. The triangles DKE and ELF are congruent by the ASA-test since $DE = EF$ (by hypothesis), and $\angle KDE = \angle LEF$ and $\angle KED = \angle LFE$ (as corresponding angles formed by a transversal intersecting parallel lines). From the congruence of the triangles, it follows that $DK = EL$. But $DK = MN$ and $EL = NP$ (as opposite sides of parallelograms), and therefore $MN = NP$.

Remark. The congruent segments can be also marked starting from the vertex of the angle B, i.e. like this: $BD = DE = EF = \dots$. Then the congruent segments on the other side of the angle are also formed starting from the vertex, i.e. $BM = MN = NP = \dots$.

94. Corollary. *The line* (DE, Figure 102) *passing through the midpoint of one side* (AB) *of a triangle and parallel to another side bisects the third side* (BC).

Indeed, on the side of the angle B, two congruent segments $BD = DA$ are marked and through the division points D and A, two parallel lines DE and AC are drawn until their intersections with the side BC. Therefore, by the theorem, the segments cut out on this side are also congruent, i.e. $BE = EC$, and thus the point E bisects BC.

Remark. The segment connecting the midpoints of two sides of a triangle is called a **midline** of this triangle.

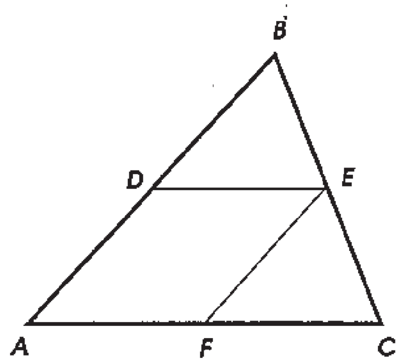

Figure 102

95. The midline theorem.

Theorem. ***The line segment*** (DE, Figure 102) ***connecting the midpoints of two sides of a triangle is parallel to the third side, and is congruent to a half of it.***

To prove this, imagine that through the midpoint D of the side AB, we draw a line parallel to the side AC. Then by the result of §94, this line bisects the side BC and thus coincides with the line DE connecting the midpoints of the sides AB and BC.

Furthermore, drawing the line $EF \| AD$, we find that the side

AC is bisected at the point F. Therefore $AF = FC$ and beside this $AF = DE$ (as opposite sides of the parallelogram $ADEF$). This implies: $DE = \frac{1}{2}AC$.

96. The trapezoid. A quadrilateral which has two opposite sides parallel and the other two opposite sides non-parallel is called a **trapezoid**. The parallel sides (AD and BC, Figure 103) of a trapezoid are called its **bases**, and the non-parallel sides (AB and CD) its **lateral sides**. If the lateral sides are congruent, the trapezoid is called **isosceles**.

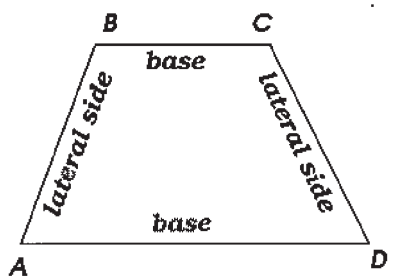

Figure 103

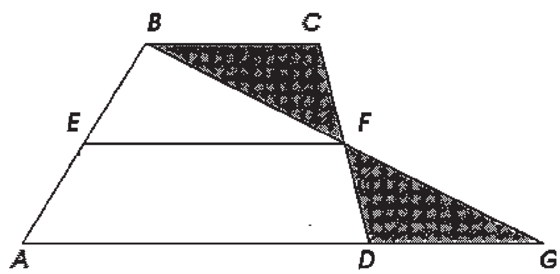

Figure 104

97. The midline of a trapezoid. The line segment connecting the midpoints of the lateral sides of a trapezoid is called its **midline**.

Theorem. ***The midline*** (EF, Figure 104) ***of a trapezoid is parallel to the bases and is congruent to their semisum.***

Through the points B and F, draw a line until its intersection with the extension of the side AD at some point G. We obtain two triangles: BCF and GDF, which are congruent by the ASA-test since $CF = FD$ (by hypothesis), $\angle BFC = \angle GFD$ (as vertical angles), and $\angle BCF = \angle GDF$ (as alternate interior angles formed by a transversal intersecting parallel lines). From the congruence of the triangles, it follows that $BF = FG$ and $BC = DG$. We see now that in the triangle ABG, the line segment EF connects the midpoints of two sides. Therefore (§95) we have: $EF \| AG$ and $EF = \frac{1}{2}(AD+DG)$, or in other words, $EF \| AD$ and $EF = \frac{1}{2}(AD + BC)$.

EXERCISES

174. Is a parallelogram considered a trapezoid?

175. How many centers of symmetry can a polygon have?

176. Can a polygon have two parallel axes of symmetry?

177. How many axes of symmetry can a quadrilateral have?

Prove theorems:

178. Midpoints of the sides of a quadrilateral are the vertices of a parallelogram. Determine under what conditions this parallelogram will be (a) a rectangle, (b) a rhombus, (c) a square.

179. In a right triangle, the median to the hypotenuse is congruent to a half of it.
Hint: Double the median by extending it past the hypotenuse.

180. Conversely, if a median is congruent to a half of the side it bisects, then the triangle is right.

181. In a right triangle, the median and the altitude drawn to the hypotenuse make an angle congruent to the difference of the acute angles of the triangle.

182. In $\triangle ABC$, the bisector of the angle A meets the side BC at the point D; the line drawn from D and parallel to CA meets AB at the point E; the line drawn from E and parallel to BC meets AC at F. Prove that $EA = FC$.

183. Inside a given angle, another angle is constructed such that its sides are parallel to the sides of the given one and are the same distance away from them. Prove that the bisector of the constructed angle lies on the bisector of the given angle.

184. The line segment connecting any point on one base of a trapezoid with any point on the other base is bisected by the midline of the trapezoid.

185. The segment between midpoints of the diagonals of a trapezoid is congruent to the semidifference of the bases.

186. Through the vertices of a triangle, the lines parallel to the opposite sides are drawn. Prove that the triangle formed by these lines consists of four triangles congruent to the given one, and that each of its sides is twice the corresponding side of the given triangle.

187. In an isosceles triangle, the sum of the distances from each point of the base to the lateral sides is constant, namely it is congruent to the altitude dropped to a lateral side.

188. How does this theorem change if points on the extension of the base are taken instead?

189. In an equilateral triangle, the sum of the distances from an interior point to the sides of this triangle does not depend on the point, and is congruent to the altitude of the triangle.

190. A parallelogram whose diagonals are congruent is a rectangle.

191. A parallelogram whose diagonals are perpendicular to each other is a rhombus.

192. Any parallelogram whose angle is bisected by the diagonal is a rhombus.

193. From the intersection point of the diagonals of a rhombus, perpendiculars are dropped to the sides of the rhombus. Prove that the feet of these perpendiculars are vertices of a rectangle.

194. Bisectors of the angles of a rectangle cut out a square.

195. Let A', B', C', and D' be the midpoints of the sides CD, DA, AB, and BC of a square. Prove that the segments AA', CC', DD', and BB' cut out a square, whose sides are congruent to 2/5th of any of the segments.

196. Given a square $ABCD$. On its sides, congruent segments AA', BB', CC', and DD' are marked. The points A', B', C', and D' are connected consecutively by lines. Prove that $A'B'C'D'$ is a square.

Find the geometric locus of:

197. The midpoints of all segments drawn from a given point to various points of a given line.

198. The points equidistant from two given parallel lines.

199. The vertices of triangles having a common base and congruent altitudes.

Construction problems

200. Draw a line parallel to a given one and situated at a given distance from it.

201. Through a given point, draw a line such that its line segment, contained between two given lines, is bisected by the given point.

202. Through a given point, draw a line such that its line segment, contained between two given parallel lines, is congruent to a given segment.

203. Between the sides of a given angle, place a segment congruent to a given segment and perpendicular to one of the sides of the angle.

204. Between the sides of a given angle, place a segment congruent to a given segment and parallel to a given line intersecting the sides of the angle.

205. Between the sides of a given angle, place a segment congruent to a given segment and such that it cuts congruent segments on the sides of the angle.

206. In a triangle, draw a line parallel to its base and such that the line segment contained between the lateral sides is congruent to the sum of the segments cut out on the lateral sides and adjacent to the base.

14 Methods of construction and symmetries

98. Problem. *To divide a given line segment* (AB, Figure 105) *into a given number of congruent parts* (e.g. into 3).

From the endpoint A, draw a line AC that forms with AB some angle. Mark on AC, starting from the point A, three congruent segments of arbitrary length: $AD = DE = EF$. Connect the point F with B, and draw through E and D lines EN and DM parallel to FB. Then, by the results of §93, the segment AB is divided by the points M and N into three congruent parts.

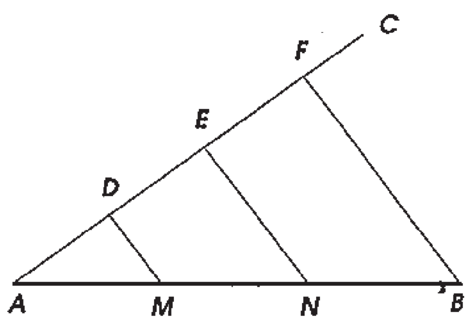

Figure 105

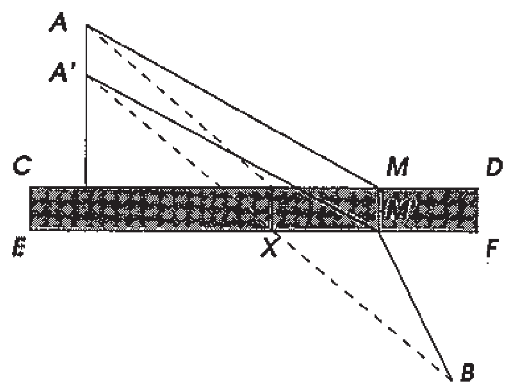

Figure 106

99. The method of parallel translation. A special method of solving construction problems, known as the method of parallel translation, is based on properties of parallelograms. It can be best explained with an example.

Problem. *Two towns A and B* (Figure 106) *are situated on opposite sides of a canal whose banks CD and EF are parallel straight lines. At which point should one build a bridge MM' across the canal in order to make the path $AM + MM' + M'B$ between the towns the shortest possible?*

To facilitate the solution, imagine that all points of the side of the canal where the town A is situated are moved downward ("translated") the same distance along the lines perpendicular to the banks of the canal as far as to make the bank CD merge with the bank EF. In particular, the point A is translated to the new position A' on the perpendicular AA' to the banks, and the segment AA' is congruent to the bridge MM'. Therefore $AA'M'M$ is a parallelogram (§86 (2)), and hence $AM = A'M'$. We conclude that the sum $AM + MM' + M'B$ is congruent to $AA' + A'M' + M'B$. The latter sum will be the shortest when the broken line $A'M'B$ is straight.

Thus the bridge should be built at that point X on bank EF where the bank intersects with the straight line $A'B$.

100. The method of reflection. Properties of axial symmetry can also be used in solving construction problems. Sometimes the required construction procedure is easily discovered when one folds a part of the diagram along a certain line (or, equivalently, **reflects** it in this line as in a mirror) so that this part occupies the symmetric position on the other side of the line. Let us give an example.

Problem. *Two towns A and B* (Figure 107) *are situated on the same side of a railroad CD which has the shape of a straight line. At which point on the railroad should one build a station M in order to make the sum $AM + MB$ of the distances from the towns to the station the smallest possible?*

Reflect the point A to the new position A' symmetric about the line CD. The segment $A'M$ is symmetric to AM about the line CD, and therefore $A'M = AM$. We conclude that the sum $AM + MB$ is congruent to $A'M + MB$. The latter sum will be the smallest when the broken line $A'MB$ is straight. Thus the station should be built at the point X where the railroad line CD intersects the straight line $A'B$.

The same construction solves yet another problem: given the line CD, and the points A and B, find a point M such that $\angle AMC = \angle BMD$.

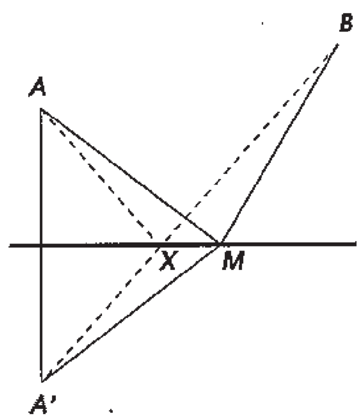

Figure 107

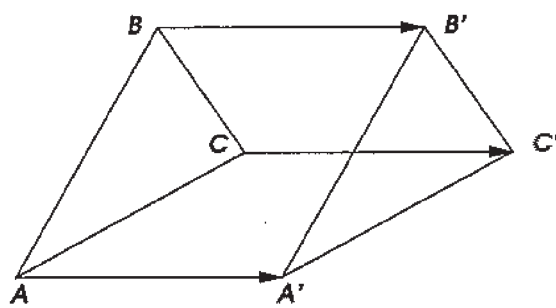

Figure 108

101. Translation. Suppose that a figure (say, a triangle ABC, Figure 108) is moved to a new position ($A'B'C'$) in a way such that all segments between the points of the figure remain parallel to themselves (i.e. $A'B' \| AB$, $B'C' \| BC$, etc.). Then the new figure is called a **translation** of the original one, and the whole motion, too, is called translation. Thus the sliding motion of a drafting triangle (Figure 76) along a straightedge (in the construction of parallel lines described in §74) is an example of translation.

Note that by the results of §86, if $AB \| A'B'$ and $AB = A'B'$ (Figure 108), then $ABB'A'$ is a parallelogram, and therefore $AA' \| BB'$ and $AA' = BB'$. Thus, if under translation of a figure, the new position A' of one point A is known, then in order to translate all other points B, C, etc., it suffices to construct the parallelograms $AA'B'B$, $AA'C'C$, etc. In other words, it suffices to construct line segments BB', CC', etc. parallel to the line segment AA', directed the same way as AA', and congruent to it.

Vice versa, if we move a figure (e.g. $\triangle ABC$) to a new position ($\triangle A'B'C'$) by constructing the line segments AA', BB', CC', etc. which are congruent and parallel to each other, and are also directed the same way, then the new figure is a translation of the old one. Indeed, the quadrilaterals $AA'B'B$, $AA'C'C$, etc. are parallelograms, and therefore all the segments AB, BC, etc. are moved to their new positions $A'B'$, $B'C'$, etc. remaining parallel to themselves.

Let us give one more example of a construction problem solved by the method of translation.

102. Problem. *To construct a quadrilateral $ABCD$ (Figure 109), given segments congruent to its sides and to the line EF connecting the midpoints of two opposite sides.*

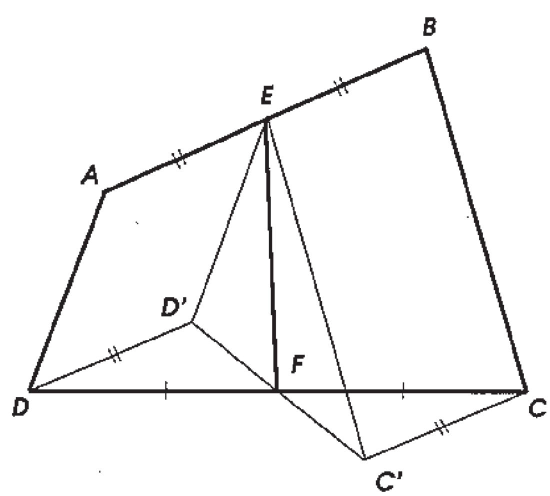

Figure 109

To bring the given lines close to each other, translate the sides AD and BC, i.e. move them in a way such that they remain parallel to themselves, to the new positions ED' and EC'. Then $DAED'$ and $C'EBC$ are parallelograms, and hence the segment DD' is congruent and parallel to AE, and the segment CC' congruent and parallel to BE. But $AE = EB$, and therefore $DD' = CC'$ and $DD' \| CC'$. As a consequence, the triangles $DD'F$ and $CC'F$ are congruent by the SAS-test (since $DD' = CC'$, $DF = FC$, and $\angle D'DF = \angle C'CF$). The congruence of the triangles implies that $\angle D'FD = \angle C'FC$, hence the broken line $D'FC'$ turns out to be straight, and therefore the figure $ED'FC'$ is a triangle. In this triangle, two sides are known ($ED' = AD$ and $EC' = BC$), and the median EF to the third side is known too. The triangle $EC'D'$ is easily recovered from these data. (Namely, double EF by extending it past F and connect the obtained endpoint with D' and C'. In the resulting parallelogram, all sides and one of the diagonals are known.)

Having recovered $\triangle ED'C'$, construct the triangles $D'DF$ and $C'CF$, and then the entire quadrilateral $ABCD$.

EXERCISES

207. Construct a triangle, given:
(a) its base, the altitude, and a lateral side;
(b) its base, the altitude, and an angle at the base;
(c) an angle, and two altitudes dropped to the sides of this angle;
(d) a side, the sum of the other two sides, and the altitude dropped to one of these sides;
(e) an angle at the base, the altitude, and the perimeter.

208. Construct a quadrilateral, given three of its sides and both diagonals.

209. Construct a parallelogram, given:
(a) two non-congruent sides and a diagonal;
(b) one side and both diagonals;
(c) the diagonals and the angle between them;
(d) a side, the altitude, and a diagonal. (Is this always possible?)

210. Construct a rectangle, given a diagonal and the angle between the diagonals.

211. Construct a rhombus, given:
(a) its side and a diagonal;
(b) both diagonals;
(c) the distance between two parallel sides, and a diagonal;
(d) an angle, and the diagonal passing through its vertex;
(e) a diagonal, and an angle opposite to it;
(f) a diagonal, and the angle it forms with one of the sides.

212. Construct a square, given its diagonal.

213. Construct a trapezoid, given:
(a) its base, an angle adjacent to it, and both lateral sides (there can be two solutions, one, or none);
(b) the difference between the bases, a diagonal, and lateral sides;
(c) the four sides (is this always possible?);
(d) a base, its distance from the other base, and both diagonals (when is this possible?);
(e) both bases and both diagonals (when is this possible?).

*214.** Construct a square, given:
(a) the sum of a diagonal and a side;
(b) the difference of a diagonal and an altitude.

*215.** Construct a parallelogram, given its diagonals and an altitude.

*216.** Construct a parallelogram, given its side, the sum of the diagonals, and the angle between them.

*217.** Construct a triangle, given:
(a) two of its sides and the median bisecting the third one;
(b) its base, the altitude, and the median bisecting a lateral side.

*218.** Construct a right triangle, given:
(a) its hypotenuse and the sum of the legs;
(b) the hypotenuse and the difference of the legs. Perform the research stage of the solutions.

219. Given an angle and a point inside it, construct a triangle with the shortest perimeter such that one of its vertices is the given point and the other two vertices lie on the sides of the angle.
`Hint:` use the method of reflection.

*220.** Construct a quadrilateral $ABCD$ whose sides are given assuming that the diagonal AC bisects the angle A.

*221.** Given positions A and B of two billiard balls in a rectangular billiard table, in what direction should one shoot the ball A so that it reflects consecutively in the four sides of the billiard and then hits the ball B?

222. Construct a trapezoid, given all of its sides.
`Hint:` use the method of translation.

*223.** Construct a trapezoid, given one of its angles, both diagonals, and the midline.

*224.** Construct a quadrilateral, given three of its sides and both angles adjacent to the unknown side.

Chapter 2

THE CIRCLE

1 Circles and chords

103. Preliminary remarks. Obviously, through a point (A, Figure 110), it is possible to draw as many circles as one wishes: their centers can be chosen arbitrarily. Through two points (A and B, Figure 111), it is also possible to draw unlimited number of circles, but their centers cannot be arbitrary since the points equidistant from two points A and B must lie on the **perpendicular bisector** of the segment AB (i.e. on the perpendicular to the segment AB passing through its midpoint, §56).

Let us find out if it is possible to draw a circle through three points.

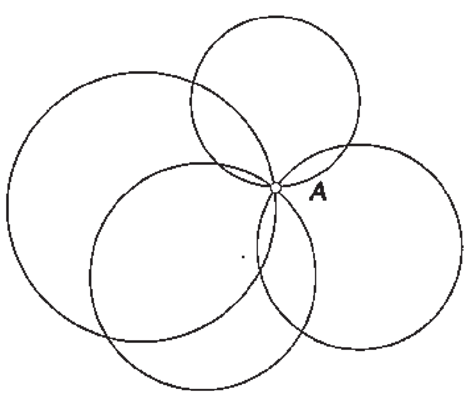

Figure 110

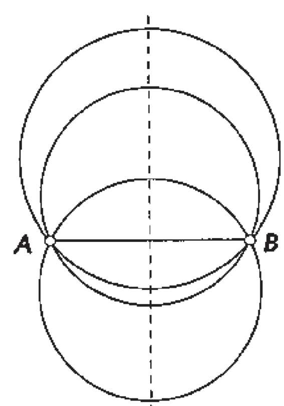

Figure 111

104. Theorem. ***Through any three points, not lying on the same line, it is possible to draw a circle, and such a circle is unique.***

Through three points A, B, C (Figure 112), not lying on the same line, (in other words, through the vertices of a triangle ABC), it is possible to draw a circle only if there exists a fourth point O, which is equidistant from the points A, B, and C. Let us prove that such a point exists and is unique. For this, we take into account that any point equidistant from the points A and B must lie on the perpendicular bisector MN of the side AB (§56). Similarly, any point equidistant from the points B and C must lie on the perpendicular bisector PQ of the side BC. Therefore, if a point equidistant from the three points A, B, and C exists, it must lie on both MN and PQ, which is possible only when it coincides with the intersection point of these two lines. The lines MN and PQ do intersect (since they are perpendicular to the intersecting lines AB and BC, §78). The intersection point O will be equidistant from A, B, and C. Thus, if we take this point for the center, and take the segment OA (or OB, or OC) for the radius, then the circle will pass through the points A, B, and C. Since the lines MN and PQ can intersect only at one point, the center of such a circle is unique. The length of the radius is also unambiguous, and therefore the circle in question is unique.

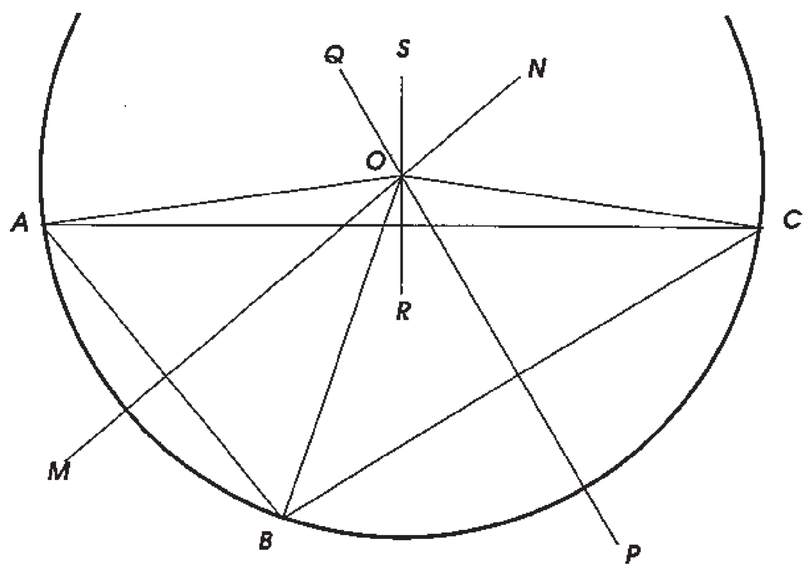

Figure 112

Remarks. (1) If the points A, B, and C (Figure 112) lay on the same line, then the perpendiculars MN and PQ would have been parallel, and therefore could not intersect. Thus, through three points lying on the same line, it is not possible to draw a circle.

(2) Three or more points lying on the same line are often called **collinear**.

Corollary. The point O, being the same distance away from A and C, has to also lie on the perpendicular bisector RS of the side AC. Thus: *three perpendicular bisectors of the sides of a triangle intersect at one point.*

105. Theorem. ***The diameter*** (AB, Figure 113), ***perpendicular to a chord, bisects the chord and each of the two arcs subtended by it.***

Fold the diagram along the diameter AB so that the left part of the diagram falls onto the right one. Then the left semicircle will be identified with the right semicircle, and the perpendicular KC will merge with KD. It follows that the point C, which is the intersection of the semicircle and KC, will merge with D. Therefore $KC = KD$, $\overset{\frown}{BC}=\overset{\frown}{BD}$, $\overset{\frown}{AC}=\overset{\frown}{AD}$.

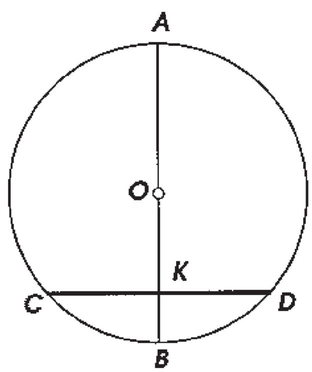

Figure 113

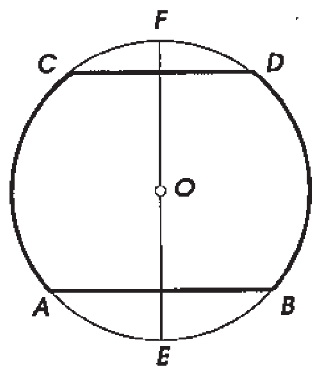

Figure 114

106. Converse theorems. (1) ***The diameter*** (AB), ***bisecting a chord*** (CD), ***is perpendicular to this chord and bisects the arc subtended by it*** (Figure 113).

(2) ***The diameter*** (AB), ***bisecting an arc*** (CBD), ***is perpendicular to the chord subtending the arc, and bisects it.***

Both propositions are easily proved by *reductio ad absurdum.*

107. Theorem. ***The arcs*** (AC and BD, Figure 114) ***contained between parallel chords*** (AB and CD) ***are congruent.***

Fold the diagram along the diameter $EF \perp AB$. Then we can conclude on the basis of the previous theorem that the point A merges with B, and the point C with D. Therefore the arc AC is identified with the arc BD, i.e. these arcs are congruent.

108. Problems. (1) *To bisect a given arc* (AB, Figure 115).

Connecting the ends of the arc by the chord AB, drop the perpendicular to this chord from the center and extend it up to the intersection point with the arc. By the result of §106, the arc AB is bisected by this perpendicular.

However, if the center is unknown, then one should erect the perpendicular to the chord at its midpoint.

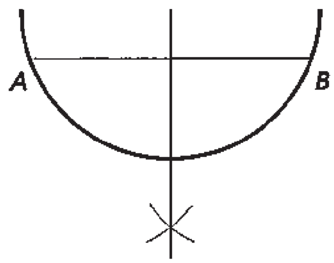

Figure 115

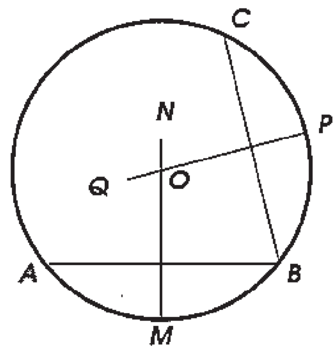

Figure 116

(2) *To find the center of a given circle* (Figure 116).

Pick on the circle any three points A, B, and C, and draw two chords through them, for instance, AB and BC. Erect perpendiculars MN and PQ to these chords at their midpoints. The required center, being equidistant from A, B, and C, has to lie on MN and PQ. Therefore it is located at the intersection point O of these perpendiculars.

109. Relationships between arcs and chords.

Theorems. ***In a disk, or in congruent disks:***

(1) ***if two arcs are congruent, then the chords subtending them are congruent and equidistant from the center;***

(2) ***if two arcs, which are smaller than the semicircle, are not congruent, then the greater of them is subtended by the greater chord, and the greater of the two chords is closer to the center.***

(1) Let an arc AB (Figure 117) be congruent to the arc CD; it is required to prove that the chords AB and CD are congruent, and that the perpendiculars OE and OF to the chords dropped from the center are congruent too.

Rotate the sector AOB about the center O so that the radius OA coincides with the radius OC. Then the arc AB will go along the arc CD, and since the arcs are congruent they will coincide. Therefore the chord AB will coincide with the chord CD, and the perpendicular OE will merge with OF (since the perpendicular from a given point to a given line is unique), i.e. $AB = CD$ and $OE = OF$.

(2) Let the arc AB (Figure 118) be smaller than the arc CD, and let both arcs be smaller than the semicircle; it is required to prove that the chord AB is smaller than the chord CD, and that the perpendicular OE is greater than the perpendicular OF.

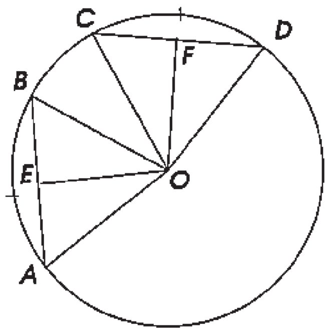

Figure 117

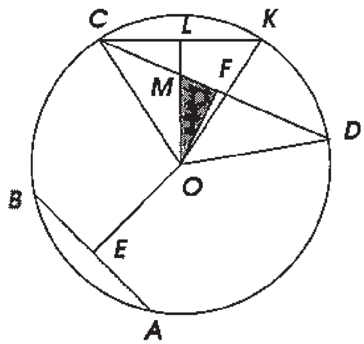

Figure 118

Mark on the arc CD the arc CK congruent to the arc AB and draw the auxiliary chord CK, which by the result of part (1) is congruent to and is the same distance away from the center as the chord AB. The triangles COD and COK have two pairs of respectively congruent sides (since they are radii), and the angles contained between these sides are not congruent. In this case (§50), the greater angle (i.e. $\angle COD$) is opposed by the greater side. Thus $CD > CK$, and therefore $CD > AB$.

In order to prove that $OE > OF$, draw $OL \perp CK$ and take into account that $OE = OL$ by the result of part (1), and therefore it suffices to compare OF with OL. In the right triangle OFM (shaded in Figure 118), the hypotenuse OM is greater than the leg OF. But $OL > OM$, and hence $OL > OF$, i.e. $OE > OF$.

The theorem just proved for *one* disk remains true for *congruent* disks because such disks differ from one another only by their position.

110. Converse theorems. Since the previous theorems address all possible mutually exclusive cases of comparative size of two arcs of the same radius (assuming that the arcs are smaller than the semicircle), and the obtained conclusions about comparative size of subtending chords or their distances from the center are mutually exclusive too, the converse propositions have to hold true as well. Namely:

In a disk, or in congruent disks:

(1) ***congruent chords are equidistant from the center and subtend congruent arcs;***

(2) ***chords equidistant from the center are congruent and subtend congruent arcs;***

(3) ***the greater one of two non-congruent chords is closer to the center and subtends the greater arc;***

(4) ***among two chords non-equidistant to the center, the one which is closer to the center subtends the greater arc.***

These propositions are easy to prove by *reductio ad absurdum*. For instance, to prove the first of them we may argue this way. If the given chords subtended non-congruent arcs, then due to the first direct theorem the chords would have been non-congruent, which contradicts the hypothesis. Therefore congruent chords must subtend congruent arcs. But when the arcs are congruent, then by the direct theorem, the subtending chords are equidistant from the center.

111. Theorem. ***A diameter is the greatest of all chords.***

Connecting the center O with the ends of any chord AB not passing through the center (Figure 119), we obtain a triangle AOB such that the chord AB is one of its sides, and the other two sides are radii. By the triangle inequality (§48) we conclude that the chord AB is smaller than the sum of two radii, while a diameter is the sum of two radii. Thus a diameter is greater than any chord not passing through the center. But since a diameter is also a chord, one can say that diameters are the greatest of all chords.

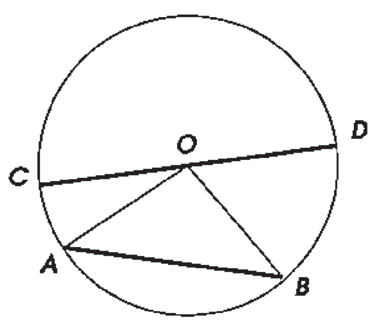

Figure 119

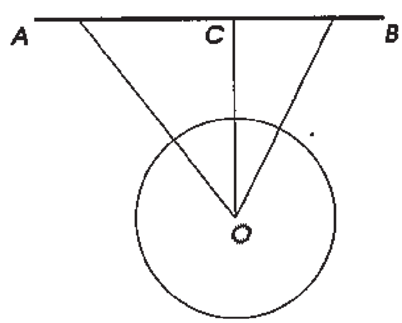

Figure 120

EXERCISES

225. A given segment is moving, remaining parallel to itself, in such a way that one of its endpoints lies on a given circle. Find the geometric locus described by the other endpoint.

226. A given segment is moving in such a way that its endpoints slide along the sides of a right angle. Find the geometric locus described by the midpoint of this segment.

227. On a chord AB, two points are taken the same distance away from the midpoint C of this chord, and through these points, two perpendiculars to AB are drawn up to their intersections with the circle. Prove that these perpendiculars are congruent.
Hint: Fold the diagram along the diameter passing through C.

228. Two intersecting congruent chords of the same circle are divided by their intersection point into respectively congruent segments.

229. In a disk, two chords CC' and DD' perpendicular to a diameter AB are drawn. Prove that the segment MM' joining the midpoints of the chords CD and $C'D'$ is perpendicular to AB.

230. Prove that the shortest of all chords, passing through a point A taken in the interior of a given circle, is the one which is perpendicular to the diameter drawn through A.

*231.** Prove that the closest and the farthest points of a given circle from a given point lie on the secant passing through this point and the center.
Hint: Apply the triangle inequality.

232. Divide a given arc into $4, 8, 16, \dots$ congruent parts.

233. Construct two arcs of the same radius, given their sum and difference.

234. Bisect a given circle by another circle centered at a given point.

235. Through a point inside a disk, draw a chord which is bisected by this point.

236. Given a chord in a disk, draw another chord which is bisected by the first one and makes a given angle with it. (Find out for which angles this is possible.)

237. Construct a circle, centered at a given point, which cuts off a chord of a given length from a given line.

238. Construct a circle of a given radius, with the center lying on one side of a given angle, and such that on the other side of the angle it cuts out a chord of a given length.

2 Relative positions of a line and a circle

112. A line and a circle can obviously be found only in one of the following mutual positions:

(1) *The distance from the center to the line is greater than the radius of the circle* (Figure 120), i.e. the perpendicular OC dropped

to the line from the center O is greater than the radius. Then the point C of the line is farther away from the center than the points of the circle and lies therefore outside the disk. Since all other points of the line are even farther away from O than the point C (slants are grater than the perpendicular), then they all lie outside the disk, and hence the line has no common points with the circle.

(2) *The distance from the center to the line is smaller than the radius* (Figure 121). In this case the point C lies inside the disk, and therefore the line and the circle intersect.

(3) *The distance from the center to the line equals the radius* (Figure 122), i.e. the point C is on the circle. Then any other point D of the line, being farther away from O than C, lies outside the disk. In this case the line and the circle have therefore only one common point, namely the one which is the foot of the perpendicular dropped from the center to the line.

Such a line, which has only one common point with the circle, is called a **tangent** to the circle, and the common point is called the **tangency point**.

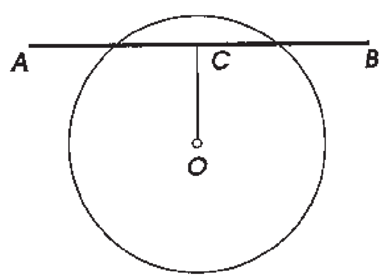

Figure 121

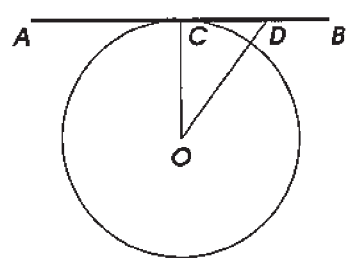

Figure 122

113. We see therefore that out of three possible cases of disposition of a line and a circle, tangency takes place only in the third case, i.e. when the perpendicular to the line dropped from the center is a radius, and in this case the tangency point is the endpoint of the radius lying on the circle. This can be also expressed in the following way:

(1) ***if a line*** (AB) ***is perpendicular to the radius*** (OC) ***at its endpoint*** (C) ***lying on the circle, then the line is tangent to the circle***, and *vice versa*:

(2) ***if a line is tangent to a circle, then the radius drawn to the tangency point is perpendicular to the line.***

114. Problem. *To construct a tangent to a given circle such that it is parallel to a given line* AB (Figure 123).

Drop to AB the perpendicular OC from the center, and through the point D, where the perpendicular intersects the circle, draw $EF \| AB$. The required tangent is EF. Indeed, since $OC \perp AB$ and $EF \| AB$, we have $EF \perp OD$, and a line perpendicular to a radius at its endpoint lying on the circle, is a tangent.

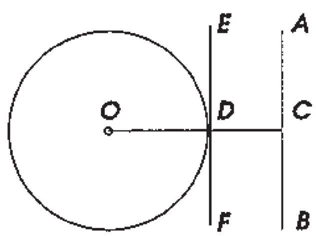

Figure 123

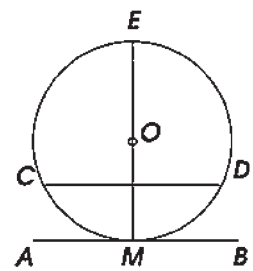

Figure 124

115. Theorem. ***If a tangent is parallel to a chord, then the tangency point bisects the arc subtended by the chord.***

Let a line AB be tangent to a circle at a point M (Figure 124) and be parallel to a chord CD; it is required to prove that $\overset{\frown}{CM} = \overset{\frown}{MD}$.

The diameter ME passing through the tangency point M is perpendicular to AB and therefore perpendicular to CD. Thus the diameter bisects the arc CMD (§105), i.e. $\overset{\frown}{CM} = \overset{\frown}{MD}$.

EXERCISES

239. Find the geometric locus of points from which the tangents drawn to a given circle are congruent to a given segment.

240. Find the geometric locus of centers of circles described by a given radius and tangent to a given line.

241. Two lines passing through a point M are tangent to a circle at the points A and B. The radius OB is extended past B by the segment $BC = OB$. Prove that $\angle AMC = 3\angle BMC$.

242. Two lines passing through a point M are tangent to a circle at the points A and B. Through a point C taken on the smaller of the arcs AB, a third tangent is drawn up to its intersection points D and E with MA and MB respectively. Prove that (1) the perimeter of $\triangle DME$, and (2) the angle DOE (where O is the center of the circle) do not depend on the position of the point C.
Hint: The perimeter is congruent to $MA+MB$; $\angle DOE = \frac{1}{2}\angle AOB$.

243. On a given line, find a point closest to a given circle.

244. Construct a circle which has a given radius and is tangent to a given line at a given point.

245. Through a given point, draw a circle tangent to a given line at another given point.

246. Through a given point, draw a circle that has a given radius and is tangent to a given line.

247. Construct a circle tangent to the sides of a given angle, and to one of them at a given point.

248. Construct a circle tangent to two given parallel lines and passing through a given point lying between the lines.

249. On a given line, find a point such that the tangents drawn from this point to a given circle are congruent to a given segment.

3 Relative positions of two circles

116. Definitions. Two circles are called **tangent** to each other if they have only one common point. Two circles which have two common points are said to **intersect** each other.

Two circles cannot have three common points since if they did, there would exist two circles passing through the same three points, which is impossible (§104).

We will call the **line of centers** the infinite line passing through the centers of two circles.

117. Theorem. ***If two circles*** (Figure 125) ***have a common point*** (A) ***situated outside the line of centers, then they have one more common point*** (A') ***symmetric to the first one with respect to the line of centers,*** (and hence such circles intersect).

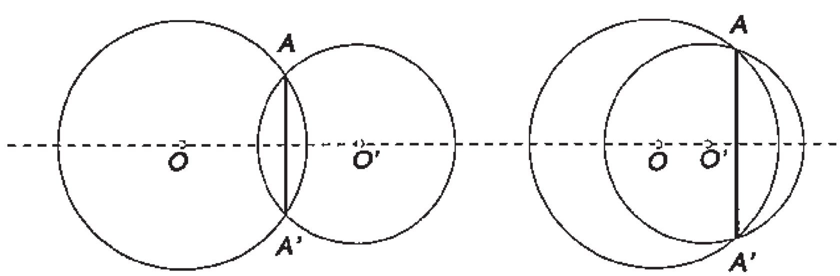

Figure 125

Indeed, the line of centers contains diameters of each of the circles and is therefore an axis of symmetry of each of them. Thus the point