3 Mathematical propositions

28. Theorems, axioms, definitions. From what we have said so far one can conclude that some geometric statements we consider quite obvious (for example, the properties of planes and lines in §3 and §4) while some others are established by way of reasoning (for example, the properties of supplementary angles in §22 and vertical angles in §26). In geometry, this process of reasoning is a principal way to discover properties of geometric figures. It would be instructive therefore to acquaint yourself with the forms of reasoning usual in geometry.

All facts established in geometry are expressed in the form of propositions. These propositions are divided into the following types.

Definitions. Definitions are propositions which explain what meaning one attributes to a name or expression. For instance, we have already encountered the definitions of central angle, right angle, perpendicular lines, etc.

Axioms. Axioms [2] are those facts which are accepted without proof. This includes, for example, some propositions we encountered earlier (§4): through any two points there is a unique line; if two points of a line lie in a given plane then all points of this line lie in the same plane.

Let us also mention the following axioms which apply to any kind of quantities:

if each of two quantities is equal to a third quantity, then these two quantities are equal to each other;

if the same quantity is added to or subtracted from equal quantities, then the equality remains true;

if the same quantity is added to or subtracted from unequal quantities, then the inequality remains unchanged, i.e. the greater quantity remains greater.

Theorems. Theorems are those propositions whose truth is found only through a certain reasoning process (proof). The following propositions may serve as examples:

if in one circle or two congruent circles some central angles are congruent, then the corresponding arcs are congruent;

if one of the four angles formed by two intersecting lines turns out to be right, then the remaining three angles are right as well.

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[2] In geometry, some axioms are traditionally called **postulates**.

Corollaries. Corollaries are those propositions which follow directly from an axiom or a theorem. For instance, it follows from the axiom "there is only one line passing through two points" that "two lines can intersect at one point at most."

29. The content of a theorem. In any theorem one can distinguish two parts: the hypothesis and the conclusion. The **hypothesis** expresses what is considered given, the **conclusion** what is required to prove. For example, in the theorem "if central angles are congruent, then the corresponding arcs are congruent" the hypothesis is the first part of the theorem: "if central angles are congruent," and the conclusion is the second part: "then the corresponding arcs are congruent;" in other words, it is given (known to us) that the central angles are congruent, and it is required to prove that under this hypothesis the corresponding arcs are congruent.

The hypothesis and the conclusion of a theorem may sometimes consist of several separate hypotheses and conclusions; for instance, in the theorem "if a number is divisible by 2 and by 3, then it is divisible by 6," the hypothesis consists of two parts: "if a number is divisible by 2" and "if the number is divisible by 3."

It is useful to notice that any theorem can be rephrased in such a way that the hypothesis will begin with the word "if," and the conclusion with the word "then." For example, the theorem "vertical angles are congruent" can be rephrased this way: "*if* two angles are vertical, *then* they are congruent."

30. The converse theorem. The theorem converse to a given theorem is obtained by replacing the hypothesis of the given theorem with the conclusion (or some part of the conclusion), and the conclusion with the hypothesis (or some part of the hypothesis) of the given theorem. For instance, the following two theorems are converse to each other:

If central angles are congruent, then the corresponding arcs are congruent.	If arcs are congruent, then the corresponding central angles are congruent.

If we call one of these theorems **direct**, then the other one should be called **converse**.

In this example both theorems, the direct and the converse one, turn out to be true. This is not always the case. For example the theorem: "if two angles are vertical, then they are congruent" is true, but the converse statement: "if two angles are congruent, then they are vertical" is false.

Indeed, suppose that in some angle the bisector is drawn (Figure 13). It divides the angle into two smaller ones. These smaller angles are congruent to each other, but they are not vertical.

EXERCISES

42. Formulate definitions of supplementary angles (§22) and vertical angles (§26) using the notion of *sides* of an angle.

43. Find in the text the definitions of an angle, its vertex and sides, in terms of the notion of a *ray drawn from a point.*

*44.** In Introduction, find the definitions of a ray and a straight segment in terms of the notions of a *straight line* and a point. Are there definitions of a point, line, plane, surface, geometric solid? Why?
Remark: These are examples of geometric notions which are considered **undefinable.**

45. Is the following proposition from §6 a definition, axiom or theorem: "Two segments are congruent if they can be laid one onto the other so that their endpoints coincide"?

46. In the text, find the definitions of a geometric figure, and congruent geometric figures. Are there definitions of congruent segments, congruent arcs, congruent angles? Why?

47. Define a circle.

48. Formulate the proposition converse to the theorem: "If a number is divisible by 2 and by 3, then it is divisible by 6." Is the converse proposition true? Why?

49. In the proposition from §10: "Two arcs of the same circle are congruent if they can be aligned so that their endpoints coincide," separate the hypothesis from the conclusion, and state the converse proposition. Is the converse proposition true? Why?

50. In the theorem: "Bisectors of supplementary angles are perpendicular," separate the hypothesis from the conclusion, and formulate the converse proposition. Is the converse proposition true?

51. Give an example that disproves the proposition: "If the bisectors of two angles with a common vertex are perpendicular, then the angles are supplementary." Is the converse proposition true?

4 Polygons and triangles

31. Broken lines. Straight segments not lying on the same line are said to form a **broken line** (Figures 31, 32) if the endpoint of the