2 Perpendicular lines

21. Right, acute and obtuse angles. An angle of 90° (congruent therefore to one half of the straight angle or to one quarter of the full angle) is called a **right angle**. An angle smaller than the right one is called **acute**, and a greater than right but smaller than straight is called **obtuse** (Figure 20).

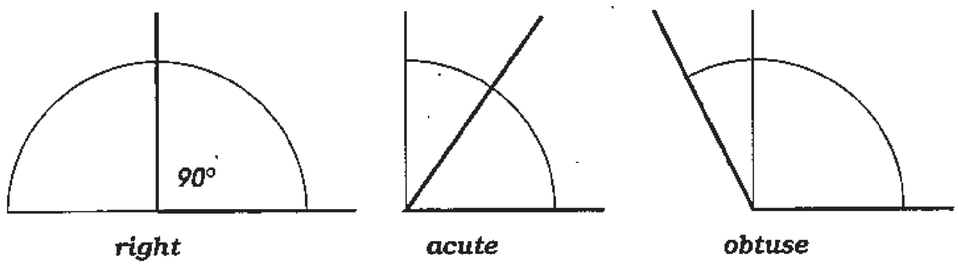

Figure 20

All right angles are, of course, *congruent to each other* since they contain the same number of degrees.

The measure of a right angle is sometimes denoted by d (the initial letter of the French word *droit* meaning "right").

22. Supplementary angles. Two angles (AOB and BOC, Figure 21) are called **supplementary** if they have one common side, and their remaining two sides form continuations of each other. Since the sum of such angles is a straight angle, *the sum of two supplementary angles is* 180° (in other words it is congruent to the sum of two right angles).

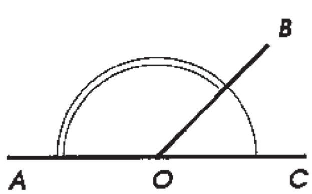

Figure 21

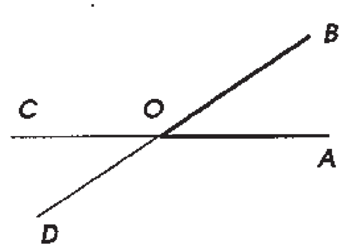

Figure 22

For each angle one can construct two supplementary angles. For example, for the angle AOB (Figure 22), prolonging the side AO we obtain one supplementary angle BOC, and prolonging the side BO we obtain another supplementary angle AOD. *Two angles supplementary to the same one are congruent to each other*, since they both contain the same number of degrees, namely the number that supplements the number of degrees in the angle AOB to 180° contained in a straight angle.

If AOB is a right angle (Figure 23), i.e. if it contains 90°, then each of its supplementary angles COB and AOD must also be right, since it contains $180° - 90°$, i.e. 90°. The fourth angle COD has to be right as well, since the three angles AOB, BOC and AOD contain 270° altogether, and therefore what is left from 360° for the fourth angle COD is 90° too. Thus, *if one of the four angles formed by two intersecting lines* (AC and BD, Figure 23) *is right, then the other three angles must be right as well.*

23. A perpendicular and a slant. In the case when two supplementary angles are not congruent to each other, their common side (OB, Figure 24) is called a **slant** [1] to the line (AC) containing the other two sides. When, however, the supplementary angles are congruent (Figure 25) and when, therefore, each of the angles is right, the common side is called a **perpendicular** to the line containing the other two sides. The common vertex (O) is called the **foot of the slant** in the first case, and the **foot of the perpendicular** in the second.

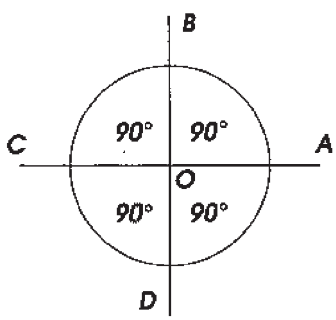

Figure 23

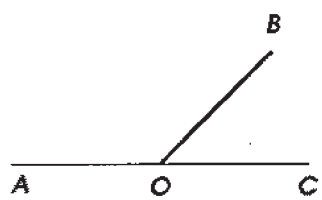

Figure 24

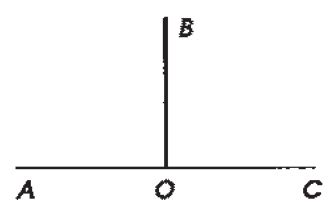

Figure 25

Two lines (AC and BD, Figure 23) intersecting at a right angle are called **perpendicular** to each other. The fact that the line AC is perpendicular to the line BD is written: $AC \perp BD$.

Remarks. (1) If a perpendicular to a line AC (Figure 25) needs to be drawn through a point O lying on this line, then the perpendicular is said to be "erected" to the line AC, and if the perpendicular needs to be drawn through a point B lying outside the line, then the perpendicular is said to be "dropped" to the line (no matter if it is upward, downward or sideways).

[1] Another name used for a slant is an **oblique line**.

(2) Obviously, at any given point of a given line, on either side of it, one can erect a perpendicular, and such a perpendicular is unique.

24. Let us prove that ***from any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.***

Let a line AB (Figure 26) and an arbitrary point M outside the line be given. We need to show that, first, one can drop a perpendicular from this point to AB, and second, that there is only one such perpendicular.

Imagine that the diagram is folded so that the upper part of it is identified with the lower part. Then the point M will take some position N. Mark this position, unfold the diagram to the initial form and then connect the points M and N by a line. Let us show now that the resulting line MN is perpendicular to AB, and that any other line passing through M, for example MD, is not perpendicular to AB. For this, fold the diagram again. Then the point M will merge with N again, and the points C and D will remain in their places. Therefore the line MC will be identified with NC, and MD with ND. It follows that $\angle MCB = \angle BCN$ and $\angle MDC = \angle CDN$.

But the angles MCB and BCN are supplementary. Therefore each of them is right, and hence $MN \perp AB$. Since MDN is not a straight line (because there can be no two straight lines connecting the points M and N), then the sum of the two congruent angles MDC and CDN is not equal to $2d$. Therefore the angle MDC is not right, and hence MD is not perpendicular to AB. Thus one can drop no other perpendicular from the point M to the line AB.

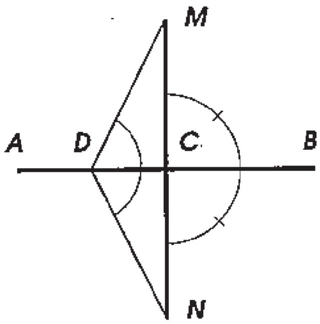

Figure 26

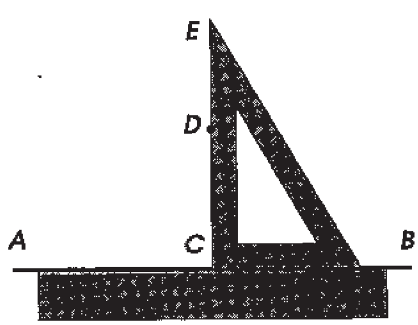

Figure 27

25. The drafting triangle. For practical construction of a perpendicular to a given line it is convenient to use a **drafting triangle** made to have one of its angles right. To draw the perpendicular to a line AB (Figure 27) through a point C lying on this line, or through a point D taken outside of this line, one can align a straightedge with the line AB, the drafting triangle with the straightedge, and then slide the triangle along the straightedge until the other side of the right angle hits the point C or D, and then draw the line CE.

26. Vertical angles. Two angles are called **vertical** if the sides of one of them form continuations of the sides of the other. For instance, at the intersection of two lines AB and CD (Figure 28) two pairs of vertical angles are formed: AOD and COB, AOC and DOB (and four pairs of supplementary angles).

Two vertical angles are congruent to each other (for example, $\angle AOD = \angle BOC$) since each of them is supplementary to the same angle (to $\angle DOB$ or to $\angle AOC$), and such angles, as we have seen (§22), are congruent to each other.

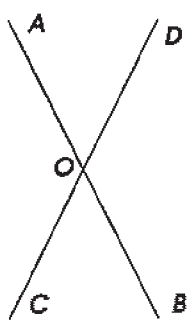

Figure 28

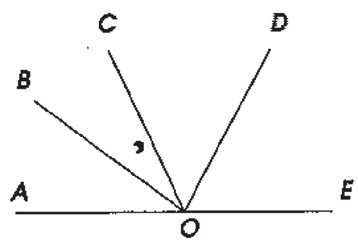

Figure 29

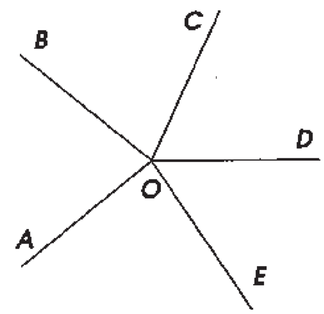

Figure 30

27. Angles that have a common vertex. It is useful to remember the following simple facts about angles that have a common vertex:

(1) *If the sum of several angles* (AOB, BOC, COD, DOE, Figure 29) *that have a common vertex is congruent to a straight angle, then the sum is* $2d$, *i.e.* $180°$.

(2) *If the sum of several angles* (AOB, BOC, COD, DOE, EOA, Figure 30) *that have a common vertex is congruent to the full angle, then it is* $4d$, *i.e.* $360°$.

(3) *If two angles* (AOB and BOC, Figure 24) *have a common vertex* (O) *and a common side* (OB) *and add up to* $2d$ (i.e. $180°$), *then their two other sides* (AO and OC) *form continuations of each other* (i.e. such angles are supplementary).

EXERCISES

28. Is the sum of the angles $14°24'44''$ and $75°35'25''$ acute or obtuse?

29. Five rays drawn from the same point divide the full angle into five congruent parts. How many different angles do these five rays form? Which of these angles are congruent to each other? Which of them are acute? Obtuse? Find the degree measure of each of them.

30. Can both angles, whose sum is the straight angle, be acute? obtuse?

31. Find the smallest number of acute (or obtuse) angles which add up to the full angle.

32. An angle measures $38°20'$; find the measure of its supplementary angles.

33. One of the angles formed by two intersecting lines is $2d/5$. Find the measures of the other three.

34. Find the measure of an angle which is congruent to twice its supplementary one.

35. Two angles ABC and CBD having the common vertex B and the common side BC are positioned in such a way that they do not cover one another. The angle $ABC = 100°20'$, and the angle $CBD = 79°40'$. Do the sides AB and BD form a straight line or a bent one?

36. Two distinct rays, perpendicular to a given line, are erected at a given point. Find the measure of the angle between these rays.

37. In the interior of an obtuse angle, two perpendiculars to its sides are erected at the vertex. Find the measure of the obtuse angle, if the angle between the perpendiculars is $4d/5$.

Prove:

38. Bisectors of two supplementary angles are perpendicular to each other.

39. Bisectors of two vertical angles are continuations of each other.

40. If at a point O of the line AB (Figure 28) two congruent angles AOD and BOC are built on the opposites sides of AB, then their sides OD and OC form a straight line.

41. If from the point O (Figure 28) rays OA, OB, OC and OD are constructed in such a way that $\angle AOC = \angle DOB$ and $\angle AOD = \angle COB$, then OB is the continuation of OA, and OD is the continuation of OC.
Hint: Apply §27, statements 2 and 3.