Chapter 1

THE STRAIGHT LINE

1 Angles

13. Preliminary concepts. A figure formed by two rays drawn from the same point is called an **angle**. The rays which form the angle are called its **sides**, and their common endpoint is called the **vertex** of the angle. One should think of the sides as extending away from the vertex indefinitely.

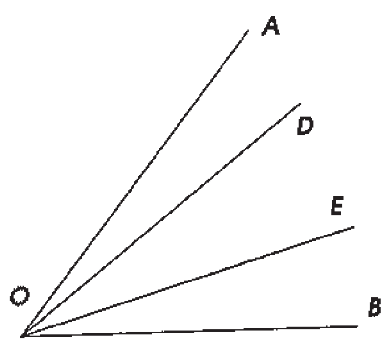

Figure 9

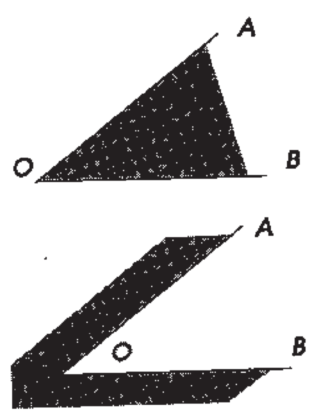

Figure 10

An angle is usually denoted by three uppercase letters of which the middle one marks the vertex, and the other two label a point on each of the sides. One says, e.g.: "the angle AOB" or "the angle BOA" (Figure 9). It is possible to denote an angle by one letter marking the vertex provided that no other angles with the same vertex are present on the diagram. Sometimes we will also denote an angle by a number placed inside the angle next to its vertex.

The sides of an angle divide the whole plane containing the angle into two regions. One of them is called the **interior** region of the angle, and the other is called the **exterior** one. Usually the interior region is considered the one that contains the segments joining any two points on the sides of the angle, e.g. the points A and B on the sides of the angle AOB (Figure 9). Sometimes however one needs to consider the other part of the plane as the interior one. In such cases a special comment will be made regarding which region of the plane is considered interior. Both cases are represented separately in Figure 10, where the interior region in each case is shaded.

Rays drawn from the vertex of an angle and lying in its interior (OD, OE, Figure 9) form new angles (AOD, DOE, EOB) which are considered to be parts of the angle (AOB).

In writing, the word "angle" is often replaced with the symbol $\angle$. For instance, instead of "angle AOB" one may write: $\angle AOB$.

14. Congruent and non-congruent angles. In accordance with the general definition of congruent figures (§1) *two angles are considered congruent if by moving one of them it is possible to identify it with the other.*

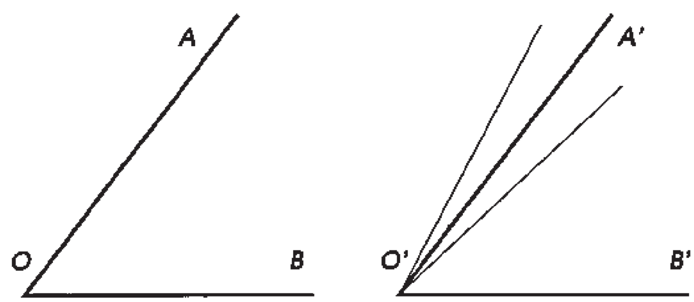

Figure 11

Suppose, for example, that we lay the angle AOB onto the angle $A'O'B'$ (Figure 11) in a way such that the vertex O coincides with O', the side OB goes along OB', and the interior regions of both angles lie on the same side of the line $O'B'$. If OA turns out to coincide with $O'A'$, then the angles are congruent. If OA turns out to lie inside or outside the angle $A'O'B'$, then the angles are non-congruent, and the one, that lies inside the other is said to be **smaller**.

15. Sum of angles. The sum of angles AOB and $A'O'B'$ (Figure 12) is an angle defined as follows. Construct an angle MNP congruent to the given angle AOB, and attach to it the angle PNQ, congruent to the given angle $A'O'B'$, as shown. Namely, the angle

MNP should have with the angle PNQ the same vertex N, a common side NP, and the interior regions of both angles should lie on the opposite sides of the common ray NP. Then the angle MNQ is called the sum of the angles AOB and $A'O'B'$. The interior region of the sum is considered the part of the plane comprised by the interior regions of the summands. This region contains the common side (NP) of the summands. One can similarly form the sum of three and more angles.

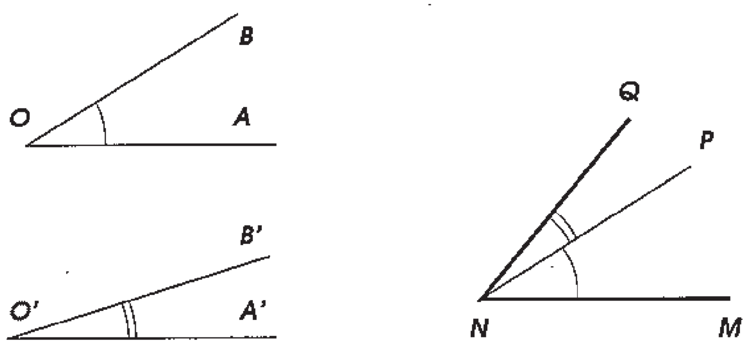

Figure 12

Addition of angles obeys the commutativity and associativity laws just the same way addition of segments does. From the concept of addition of angles one derives the concept of subtraction of angles, and multiplication and division of angles by a whole number.

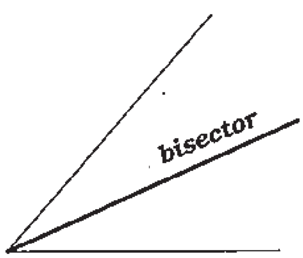

Figure 13

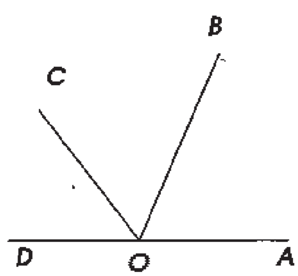

Figure 14

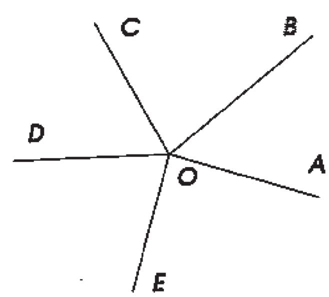

Figure 15

Very often one has to deal with the ray which divides a given angle into halves; this ray is called the **bisector** of the angle (Figure 13).

16. Extension of the concept of angle. When one computes the sum of angles some cases may occur which require special attention.

(1) It is possible that after addition of several angles, say, the

three angles: AOB, BOC and COD (Figure 14), the side OD of the angle COD will happen to be the continuation of the side OA of the angle AOB. We will obtain therefore the figure formed by two half-lines (OA and OD) drawn from the same point (O) and continuing each other. Such a figure is also considered an angle and is called a **straight angle.**

(2) It is possible that after the addition of several angles, say, the five angles: AOB, BOC, COD, DOE and EOA (Figure 15) the side OA of the angle EOA will happen to coincide with the side OA of the angle AOB. The figure formed by such rays (together with the whole plane surrounding the vertex O) is also considered an angle and is called a **full angle.**

(3) Finally, it is possible that added angles will not only fill in the whole plane around the common vertex, but will even overlap with each other, covering the plane around the common vertex for the second time, for the third time, and so on. Such an angle sum is congruent to one full angle added with another angle, or congruent to two full angles added with another angle, and so on.

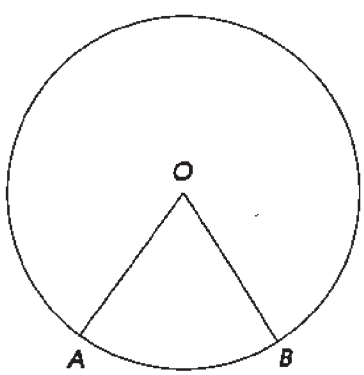

Figure 16

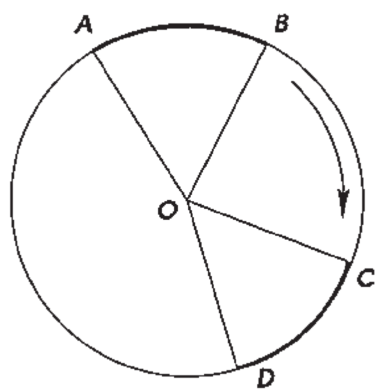

Figure 17

17. Central angle. The angle (AOB, Figure 16) formed by two radii of a circle is called a **central angle**; such an angle and the arc contained between the sides of this angle are said to *correspond* to each other.

Central angles and their corresponding arcs have the following properties.

In one circle, or two congruent circles:

(1) ***If central angles are congruent, then the corresponding arcs are congruent;***

(2) *Vice versa,* ***if the arcs are congruent, then the corre-***

sponding central angles are congruent.

Let $\angle AOB = \angle COD$ (Figure 17); we need to show that the arcs AB and CD are congruent too. Imagine that the sector AOB is rotated about the center O in the direction shown by the arrow until the radius OA coincides with OC. Then due to the congruence of the angles, the radius OB will coincide with OD; therefore the arcs AB and CD will coincide too, i.e. they are congruent.

The second property is established similarly.

18. Circular and angular degrees. Imagine that a circle is divided into 360 congruent parts and all the division points are connected with the center by radii. Then around the center, 360 central angles are formed which are congruent to each other as central angles corresponding to congruent arcs. Each of these arcs is called a **circular degree**, and each of those central angles is called an **angular degree**. Thus one can say that a circular degree is 1/360th part of the circle, and the angular degree is the central angle corresponding to it.

The degrees (both circular and angular) are further subdivided into 60 congruent parts called **minutes**, and the minutes are further subdivided into 60 congruent parts called **seconds**.

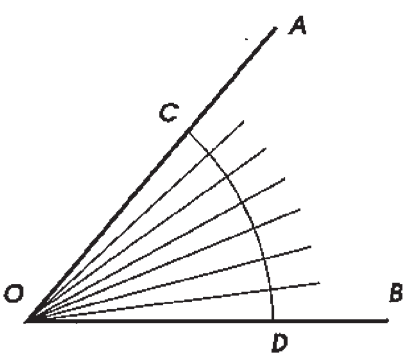

Figure 18

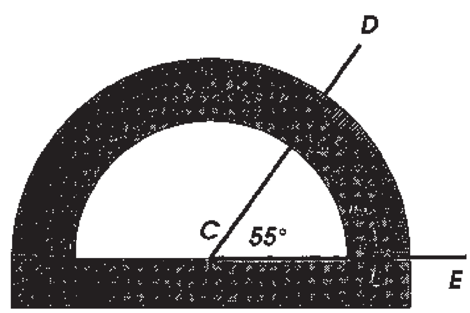

Figure 19

19. Correspondence between central angles and arcs. Let AOB be some angle (Figure 18). Between its sides, draw an arc CD of arbitrary radius with the center at the vertex O. Then the angle AOB will become the central angle corresponding to the arc CD. Suppose, for example, that this arc consists of 7 circular degrees (shown enlarged in Figure 18). Then the radii connecting the division points with the center obviously divide the angle AOB into 7 angular degrees. More generally, one can say that *an angle is measured by the arc corresponding to it,* meaning that an angle contains as many angular degrees, minutes and seconds as the corresponding arc contains circular degrees, minutes and seconds. For instance, if the arc CD contains 20 degrees 10 minutes and 15 seconds of circular units, then the angle AOB consists of 20 degrees 10 minutes and 15 seconds of angular units, which is customary to express as: $\angle AOB = 20°10'15''$, using the symbols $°$, $'$ and $''$ to denote degrees, minutes and seconds respectively.

Units of angular degree do not depend on the radius of the circle. Indeed, adding 360 angular degrees following the summation rule described in §15, we obtain the full angle at the center of the circle. Whatever the radius of the circle, this full angle will be the same. Thus one can say that an angular degree is 1/360th part of the full angle.

20. Protractor. This device (Figure 19) is used for measuring angles. It consists of a semi-disk whose arc is divided into 180°. To measure the angle DCE, one places the protractor onto the angle in a way such that the center of the semi-disk coincides with the vertex of the angle, and the radius CB lies on the side CE. Then the number of degrees in the arc contained between the sides of the angle DCE shows the measure of the angle. Using the protractor one can also draw an angle containing a given number of degrees (e.g. the angle of 90°, 45°, 30°, etc.).

EXERCISES

20. Draw any angle and, using a protractor and a straightedge, draw its bisector.

21. In the exterior of a given angle, draw another angle congruent to it. Can you do this in the interior of the given angle?

22. How many common sides can two distinct angles have?

23. Can two non-congruent angles contain 55 angular degrees each?

24. Can two non-congruent arcs contain 55 circular degrees each? What if these arcs have the same radius?

25. Two straight lines intersect at an angle containing 25°. Find the measures of the remaining three angles formed by these lines.

26. Three lines passing through the same point divide the plane into six angles. Two of them turned out to contain 25° and 55° respectively. Find the measures of the remaining four angles.

*27.** Using only compass, construct a 1° arc on a circle, if a 19° arc of this circle is given.